A Tableaux-based Algorithm for Description Logics with Transitive Closure of Roles in Concept and Role Inclusion Axioms

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Abstract. In this paper, we investigate an extension of the description logic $\mathcal{SHIQ}$—a knowledge representation formalism used for the Semantic Web—with transitive closure of roles occurring not only in concept inclusion axioms but also in role inclusion axioms. We start by proving that adding transitive closure of roles to $\mathcal{SHIQ}$ without restriction on role hierarchies may lead to undecidability. An analysis of this proof allows us to identify kinds of axioms which are responsible for the undecidability and to design a decidable extension of $\mathcal{SHIQ}$ with transitive closure of roles. Next, we propose a tableaux-based algorithm that decides satisfiability of the new logic. It was shown by experiments that this kind of algorithms is suitable for implementation.

1 Introduction

The ontology language OWL-DL [1] is widely used to formalize semantic resources on the Semantic Web. This language is mainly based on the description logic $\mathcal{SHOIN}$ which is known to be decidable [2]. Although $\mathcal{SHOIN}$ is expressive and provides *transitive roles* to model transitivity of relations, we can find several applications in which *the transitive closure of roles*, that is more expressive than transitive roles, is necessary. An example given by [3] describes two categories of devices as follows: (1) Devices have as their direct part a battery: $\mathsf{Device} \sqcap \exists \mathsf{hasPart}.\mathsf{Battery}$, (2) Devices have at some level of decomposition a battery: $\mathsf{Device} \sqcap \exists \mathsf{hasPart}^+.\mathsf{Battery}$. However, if we now define $\mathsf{hasPart}$ as a *transitive role*, the concept $\mathsf{Device} \sqcap \exists \mathsf{hasPart}.\mathsf{Battery}$ does not represent the devices as described above since it does not allow one to describe these categories of devices as two different sets of devices.

In addition, the difference between transitive roles and the transitive closure of roles is clearer when they are involved in role inclusion axioms. It is obvious that concept $\exists \mathsf{R}^+.(\mathsf{C} \sqcap \forall \mathsf{R}^-.\bot)$ is unsatisfiable w.r.t an empty TBox and the trivial axiom $\mathsf{R} \sqsubseteq \mathsf{R}^+$. If we now substitute R^+ for a transitive role R_t such that $\mathsf{R} \sqsubseteq \mathsf{R}_t$ (i.e. we substitute each occurrence of R^+ in axioms and concepts for R_t) then the concept $\exists \mathsf{R}_t.(\mathsf{C} \sqcap \forall \mathsf{R}^-.\bot)$ becomes satisfiable. The point is that an instance of R^+ represents a sequence of instances of R but an instance of R_t corresponds to a sequence of instances of *itself*.

In several applications, we need to model successive events and relationships between them. An event is something oriented in time i.e. we can talk about endpoints of an event, or a chronological order of events. When an event of some kind occurs it can trigger an event (or a sequence of events) of another kind. In this situation, it may be suitable for using a role to model an event. If we denote roles event and event' for two kinds of events then the axiom $(\mathsf{event} \sqsubseteq \mathsf{event}')$ expresses the fact that when an event of the first kind occurs it implies one event or a sequence of events of the second kind. To express "a sequence of events" we can define event' to be transitive. However, the semantics of transitive roles is not sufficient to describe this behaviour since the transitive role event' can represent a sequence of itself but not a sequence of another role. Such behaviours can be found in the following example.

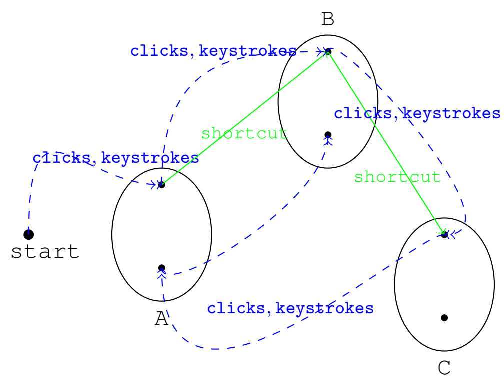

Fig. 1. Mouse clicks and keystrokes

Example 1. Let $\mathbf{S}$ be the set of all states of applications running on a computer. We denote by $\mathsf{A}, \mathsf{B}, \mathsf{C} \subseteq \mathbf{S}$ the sets of states of applications $\mathsf{A}, \mathsf{B}, \mathsf{C}$, respectively. A user can perform a mouse-click or keystroke to change states. The user can type a shortcut (combination of keys) to go from A to B or from B to C. This action corresponds to a sequence of mouse-clicks or keystrokes. The system's behaviour is depicted in Figure 1.

In such a system, users may be interested in the following question: "from the application A, can one go through the application B to get directly the application C by a mouse-click or keystroke ?".

We now use a description logic with transitive closure of roles to express the constraints as described above. To to this, we use a role next to model *mouse clicks* or *keystrokes* and a role jump to model *shortcuts* in the following axioms:

(i) $\mathsf{start} \sqsubseteq \neg\mathsf{A} \sqcap \neg\mathsf{B} \sqcap \neg\mathsf{C}$; $\mathsf{X} \sqcap \mathsf{Y} \sqsubseteq \bot$ with $\mathsf{X}, \mathsf{Y} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ and $\mathsf{X} \neq \mathsf{Y}$;

(ii) $\mathsf{A} \sqsubseteq \exists \mathsf{jump}.\mathsf{B}$; $\mathsf{A} \sqsubseteq \exists \mathsf{jump}.\mathsf{C}$; $\mathsf{B} \sqsubseteq \exists \mathsf{jump}.\mathsf{C}$;

(iii) $\mathsf{start} \sqsubseteq \forall \mathsf{next}^-.\bot$; $\mathsf{jump} \sqsubseteq \mathsf{next}^+$;

Under some operating systems, users cannot switch directly from an application to a particular one just by one mouse click or keystroke. We can express this constraint by an axiom as follows:

(iv) $\mathsf{C} \sqcap \exists \mathsf{next}^-.\mathsf{B} \sqsubseteq \bot$;

In this case, the concept $(\mathsf{A} \sqcap \exists \mathsf{next}^+.(\mathsf{C} \sqcap \exists \mathsf{next}^-.\mathsf{B}))$ capturing the question above is unsatisfiable w.r.t. the axioms presented.

Such examples motivate the study of Description Logics (DL) that allow the transitive closure of roles to occur in both concept and role inclusion axioms. We introduce in this work a DL that can model systems as described in Example 1 and propose a tableaux-based decision procedure for concept satisfiability problem in this DL.

To the best of our knowledge, the decidability of $\mathcal{SHIQ}_+$, which is obtained from $\mathcal{SHIQ}$ by adding transitive closure of roles, is unknown. [4] and [5] have established decision procedures for concept satisfiability in $\mathcal{SHI}_+$ and $\mathcal{SHIO}_+$ by using *neighborhoods* representing an individual with its neighbors in a model, to build completion graphs. In the literature, many decidability results in DLs can be obtained from their counterparts in modal logics ([6], [7]). However, these counterparts do not take into account expressive role inclusion axioms. In particular, [7] has shown decidability of a very expressive DL, so-called $\mathcal{CATS}$, including $\mathcal{SHIQ}$ with the transitive closure of roles but not allowing it to occur in role inclusion axioms. [7] has pointed out that the complexity of concept subsumption in $\mathcal{CATS}$ is EXPTIME-complete by translating $\mathcal{CATS}$ into the logic Converse PDL in which inference problems are well studied.

Recently, there have been some works in [8] and [9] which have attempted to augment the expressiveness of role inclusion axioms. A decidable logic, namely $\mathcal{SROIQ}$, resulting from these efforts allows for new role constructors such as composition, disjointness and negation. In addition, [10] has introduced a DL, so-called $\mathcal{ALCQI}b^+_{reg}$, which can capture $\mathcal{SRIQ}$ ($\mathcal{SROIQ}$ without nominal), and obtained the worst-case complexity (EXPTIME-complete) of the satisfiability problem by using automata-based technique. $\mathcal{ALCQI}b^+_{reg}$ allows for a rich set of operators on roles by which one can simulate role inclusion axioms. However, transitive closures in role inclusion axioms are expressible neither in $\mathcal{SROIQ}$ nor in $\mathcal{ALCQI}b^+_{reg}$.

In addition, tableaux-based algorithms for expressive DLs like $\mathcal{SHIQ}$ [11] and $\mathcal{SHOIQ}$ [12] result in efficient implementations. This kind of algorithms relies on two structures, the so-called *tableau* and *completion graph*. Roughly speaking, a tableau for a concept represents a model for the concept and it is possibly infinite. A tableau translates satisfiability of all given concept and role inclusion axioms into the satisfiability of semantic constraints imposed *locally* on each individual of the tableau. This feature of tableaux will be called *local satisfiability property*. In turn, a completion graph for a concept is a *finite* representation from which a tableau can be built. The algorithm in [13] for satisfiability in $\mathcal{ALC}_{reg}$ (including the transitive closure of roles and other role operators) introduced a method to deal with loops which can hide unsatisfiable nodes.

Regarding undecidability results, [9] has shown that an arbitrary extension of role inclusion axioms, such as adding $R \circ S \sqsubseteq P$, may lead to undecidability. Additionally, as it turned out by [11], the interaction between transitive roles and number restrictions causes also undecidability. The technique used to prove these undecidability results is to reduce the domino problem, which is known to be undecidable [14], to the problem in question.

Tableaux-based algorithms for expressive DLs such as $\mathcal{SHIQ}$ [11] and $\mathcal{SHOIQ}$ [12] result in efficient implementations. This kind of algorithms relies on two structures, the so-called *tableau* and *completion graph*. Roughly speaking, a tableau for a concept represents a model for the concept and it is possibly infinite. A tableau translates satisfiability of all given concept and role inclusion axioms into the satisfiability of constraints imposed *locally* on each individual of the tableau by the semantics of concepts in the individual's label. This feature of tableaux will be called *local satisfiability property*. To check satisfiability of a concept, tableaux-based algorithms try to build a completion graph whose finiteness is ensured by a technique, the so-called *blocking technique*. It provides a termination condition and guarantees soundness and completeness. The underlying idea of the blocking mechanism is to detect "loops" which are repeated pieces of a completion graph. When transitive closure of roles is added to knowledge bases, this blocking technique allows us to lengthen paths through such loops in order to satisfy semantic constraints imposed by transitive closures. The algorithm in [13] for satisfiability in $\mathcal{ALC}_{reg}$ (including the transitive closure of roles and other role operators) introduced a method to deal with loops which can hide unsatisfiable nodes. This method detects on so-called *concept trees*, "good" or "bad" cycles that are similar to those between blocking and blocked nodes on completion trees.

To deal with transitive closure of roles occurring in terms such as $\exists Q^+.C$, we have to introduce a new expansion rule to build completion trees such that it can generate a path formed from nodes that are connected by edges whose label contains role Q. In addition, this rule propagates terms $\exists Q^+.C$ to each node along with the path before reaching a node whose label includes concept C. Such a path may go through blocked and blocking nodes and has an arbitrary length. To handle transitive closures of roles occurring in role inclusion axioms such as $R \sqsubseteq Q^+$, we use another new expansion rule that translates satisfaction of such axioms into satisfaction of a term $\exists Q^+.\Phi$. From the path generated from $\exists Q^+.\Phi$, a cycle can be formed to satisfy the semantic constraint imposed by $R \sqsubseteq Q^+$. Since the role Q, which will be defined to be *simple*, does not occur in number restrictions, the cycle obtained from this way does not violate other semantic constraints.

The contribution of the present paper consists of (i) proving that if we add transitive closure of roles to $\mathcal{SHIQ}$ without restriction the obtained logic is undecidable even if roles are simple according to the defintion presented in [11], (ii) designing a new logic, namely $\mathcal{SHIQ}_+$, with a new definition for simple roles and presenting a tableaux-based algorithm for satisfiability of concepts in $\mathcal{SHIQ}_+$.

2 The Description Logic $\mathcal{SHIQ}_+$

The logic $\mathcal{SHIQ}_+$ is an extension of $\mathcal{SHIQ}$ by allowing transitive closure of roles to occur in concept and role inclusion axioms. In this section, we present the syntax and semantics of the logic $\mathcal{SHIQ}_+$. This includes the definitions of inference problems and how they are interrelated. The definitions reuse some notation introduced in [12].

Definition 1. *Let* $\mathbf{R}$ *be a non-empty set of* role names. *We denote* $\mathbf{R}_{\mathsf{I}} = \{P^- \mid P \in \mathbf{R}\}$ *and* $\mathbf{R}_+ = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_{\mathsf{I}}\}$.

$*$ *The set of* $\mathcal{SHIQ}_+$*-roles is* $\mathbf{R} \cup \mathbf{R}_{\mathsf{I}} \cup \mathbf{R}_+$. *A* role inclusion axiom *is of the form* $R \sqsubseteq S$ *for two* $\mathcal{SHIQ}_+$*-roles* R *and* S. *A* role hierarchy $\mathcal{R}$ *is a finite set of role inclusion axioms.*

$*$ *An interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *consists of a non-empty set* $\Delta^{\mathcal{I}}$ *(*domain*) and a function* $\cdot^{\mathcal{I}}$ *which maps each role name to a subset of* $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ *such that, for* $R \in \mathbf{R}$, $Q^+ \in \mathbf{R}_+$,

$R^{-\mathcal{I}} = \{\langle x, y\rangle \in (\Delta^{\mathcal{I}})^2 \mid \langle y, x\rangle \in R^{\mathcal{I}}\}$, $(Q^+)^{\mathcal{I}} = \bigcup_{n>0} (Q^n)^{\mathcal{I}}$ *with* $(Q^1)^{\mathcal{I}} = Q^{\mathcal{I}}$ *and*

$(Q^n)^{\mathcal{I}} = \{\langle x, y\rangle \in (\Delta^{\mathcal{I}})^2 \mid \exists z \in \Delta^{\mathcal{I}}, \langle x, z\rangle \in (Q^{n-1})^{\mathcal{I}}, \langle z, y\rangle \in Q^{\mathcal{I}}\}$.

An interpretation $\mathcal{I}$ *satisfies a role hierarchy* $\mathcal{R}$ *if* $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$ *for each* $R \sqsubseteq S \in \mathcal{R}$. *Such an interpretation is called a* model *of* $\mathcal{R}$*, denoted by* $\mathcal{I} \models \mathcal{R}$.

$*$ *To simplify notations for nested inverse roles and transitive closures of roles, we define two functions* $\cdot^{\ominus}$ *and* $\cdot^{\oplus}$ *as follows:*

$$R^{\ominus} = \begin{cases} R^- & \text{if } R \in \mathbf{R}, \\ S & \text{if } R = S^- \text{ and } S \in \mathbf{R}, \\ (S^-)^+ & \text{if } R = S^+ \text{ and } S \in \mathbf{R}, \\ S^+ & \text{if } R = (S^-)^+ \text{ and } S \in \mathbf{R} \end{cases} \quad R^{\oplus} = \begin{cases} R^+ & \text{if } R \in \mathbf{R}, \\ S^+ & \text{if } R = (S^+)^+ \text{ and } S \in \mathbf{R}, \\ (S^-)^+ & \text{if } R = S^- \text{ and } S \in \mathbf{R}, \\ (S^-)^+ & \text{if } R = (S^+)^- \text{ and } S \in \mathbf{R} \end{cases}$$

$*$ *A relation* $\underline{\circledast}$ *is defined as the transitive-reflexive closure* $\mathcal{R}^+$ *of* $\sqsubseteq$ *on* $\mathcal{R} \cup \{R^{\ominus} \sqsubseteq S^{\ominus} \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{R^{\oplus} \sqsubseteq S^{\oplus} \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^{\oplus} \mid Q \in \mathbf{R} \cup \mathbf{R}_{\mathsf{I}}\}$. *We denote* $S \equiv R$ *iff* $R \underline{\circledast} S$ *and* $S \underline{\circledast} R$.

$*$ *A role* R *is called* simple *w.r.t.* $\mathcal{R}$ *iff (i)* $Q^{\oplus} \underline{\circledast} R \notin \mathcal{R}^+$ *for each* $Q \in \mathbf{R} \cup \mathbf{R}_{\mathsf{I}}$*, and (ii)* $R' \underline{\circledast} R$, $P \underline{\circledast} R'^{\oplus} \in \mathcal{R}^+$ *implies* $P \underline{\circledast} R' \in \mathcal{R}^+$. *We define a function* $\mathsf{Cyc}(R)$ *which returns true iff* R *does not satisfy the condition (ii).*

The reason for the introduction of two functions $\cdot^{\ominus}$ and $\cdot^{\oplus}$ in Definition 1 is that they can avoid using R^{--} and R^{++} and it remains a unique nested case $(R^-)^+$.

Notice that a transitive role S (i.e. $\langle x, y\rangle \in S^{\mathcal{I}}, \langle y, z\rangle \in S^{\mathcal{I}}$ implies $\langle x, z\rangle \in S^{\mathcal{I}}$ where $\mathcal{I}$ is an interpretation) can be expressed by using a role axiom $S^{\oplus} \sqsubseteq S$. In addition, a role R which is simple according to Definition 1 is simple according to [11] as well. In fact, if $Q^{\oplus} \underline{\circledast} R \notin \mathcal{R}^+$ for each $Q \in \mathbf{R} \cup \mathbf{R}_{\mathsf{I}}$ then there is no transitive role S such that $S \underline{\circledast} R \in \mathcal{R}^+$. Otherwise, we have $Q^{\oplus} \underline{\circledast} R \in \mathcal{R}^+$ since $Q^{\oplus} \underline{\circledast} S \in \mathcal{R}^+$. Finally, if $R \underline{\circledast} S \in \mathcal{R}^+$ and R is not simple according to Definition 1 then S is not simple according to Definition 1.

Definition 2. *Let* $\mathbf{C}$ *be a non-empty set of* concept names.

$*$ *The set of* $\mathcal{SHIQ}_+$*-concepts is inductively defined as the smallest set containing all* C *in* $\mathbf{C}$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n\, S.C)$ *and* $(\geq n\, S.C)$ *where* C *and* D *are* $\mathcal{SHIQ}_+$*-concepts,* R *is an* $\mathcal{SHIQ}_+$*-role and* S *is a simple role. We denote* $\bot$ *for* $\neg\top$.

$*$ *An interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *consists of a non-empty set* $\Delta^{\mathcal{I}}$ *(*domain*) and a function* $\cdot^{\mathcal{I}}$ *which maps each concept name to a subset of* $\Delta^{\mathcal{I}}$ *such that*

$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$,

$(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}, \langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$,

$(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}}, \langle x,y\rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$,
$(\geq n\, S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \mathsf{card}\{y \in C^{\mathcal{I}} \mid \langle x,y\rangle \in S^{\mathcal{I}}\} \geq n\}$,
$(\leq n\, S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \mathsf{card}\{y \in C^{\mathcal{I}} \mid \langle x,y\rangle \in S^{\mathcal{I}}\} \leq n\}$
where $\mathsf{card}\{S\}$ *is denoted for the cardinality of a set* S.
$*$ $C \sqsubseteq D$ *is called a general concept inclusion (GCI) where* C, D *are* $\mathcal{SHIQ}_+$-*concepts (possibly complex), and a finite set of GCIs is called a terminology* $\mathcal{T}$. *An interpretation* $\mathcal{I}$ *satisfies a GCI* $C \sqsubseteq D$ *if* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ *and* $\mathcal{I}$ *satisfies a terminology* $\mathcal{T}$ *if* $\mathcal{I}$ *satisfies each GCI in* $\mathcal{T}$. *Such an interpretation is called a* model *of* $\mathcal{T}$, *denoted by* $\mathcal{I} \models \mathcal{T}$.

$*$ *A concept* C *is called* satisfiable *w.r.t. a role hierarchy* $\mathcal{R}$ *and a terminology* $\mathcal{T}$ *iff there is some interpretation* $\mathcal{I}$ *such that* $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ *and* $C^{\mathcal{I}} \neq \emptyset$. *Such an interpretation is called a* model *of* C *w.r.t.* $\mathcal{R}$ *and* $\mathcal{T}$. *A pair* $(\mathcal{T}, \mathcal{R})$ *is called a* $\mathcal{SHIQ}_+$ *knowledge base and said to be consistent if there is a model* $\mathcal{I}$ *of both* $\mathcal{T}$ *and* $\mathcal{R}$, *i.e.,* $\mathcal{I} \models \mathcal{T}$ *and* $\mathcal{I} \models \mathcal{R}$.

$*$ *A concept* D subsumes *a concept* C *w.r.t.* $\mathcal{R}$ *and* $\mathcal{T}$, *denoted by* $C \sqsubseteq D$, *if* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ *holds in each model* $\mathcal{I}$ *of* $(\mathcal{T}, \mathcal{R})$.

Since negation is allowed in the logic $\mathcal{SHIQ}_+$, unsatisfiability and subsumption w.r.t. $(\mathcal{T}, \mathcal{R})$ can be reduced each other: $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable. In addition, we can reduce ontology consistency to concept satisfiability w.r.t. a knowledge base : $(\mathcal{T}, \mathcal{R})$ is consistent if $A \sqcup \neg A$ is satisfiable w.r.t. $(\mathcal{T}, \mathcal{R})$ for some concept name A. Thanks to these reductions, it suffices to study the satisfiability of a concept C w.r.t. a knowledge base $(\mathcal{T}, \mathcal{R})$.

For the ease of construction, we assume all concepts to be in *negation normal form* (NNF) i.e. negation occurs only in front of concept names. Any $\mathcal{SHIQ}_+$-concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in [11]. For a concept C, we denote the nnf of C by $\mathsf{nnf}(C)$ and the nnf of $\neg C$ by $\dot{\neg} C$

Let D be an $\mathcal{SHIQ}_+$-concept in NNF. We define $\mathsf{sub}(D)$ to be the smallest set that contains all sub-concepts of D including D. In addition,

For a knowledge base $(\mathcal{T}, \mathcal{R})$, we use $\mathbf{R}_{(\mathcal{T},\mathcal{R})}$ to denote the set of all role names occurring in $\mathcal{T}, \mathcal{R}$ with their transitive closure and inverses. We denote by $\mathbf{R}^{+}_{(\mathcal{T},\mathcal{R})}$ the set of transitive closure of roles occuring in $\mathbf{R}_{(\mathcal{T},\mathcal{R})}$. In addition, we define sets $\mathsf{sub}(\mathcal{T}, \mathcal{R})$

and $\widehat{\mathsf{sub}}(\mathcal{T},\mathcal{R})$ as follows:

$$
\begin{aligned}
\mathsf{sub}(\mathcal{T},\mathcal{R}) &= \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(\mathsf{nnf}(\neg C \sqcup D),\mathcal{R}) \textit{ where} && (1)\\
\mathsf{sub}(E,\mathcal{R}) &= \mathsf{sub}(E) \cup \{\dot{\neg} C \mid C \in \mathsf{sub}(E)\} \; \cup && (2)\\
&\quad \{\forall S.C \mid (\forall R.C \in \mathsf{sub}(E), S \underline{\boxast} R) \textit{ or } (\dot{\neg}\forall R.C \in \mathsf{sub}(E), S \underline{\boxast} R) \\
&\qquad \textit{where } S \textit{ occurs in } \mathcal{T} \textit{ or } \mathcal{R}\} \; \cup \\
&\quad \{\exists P.\beta \mid \beta \in \{C, \exists P^{\oplus}.C\}, \exists P^{\oplus}.C \in \mathsf{sub}(E)\} \\
\Phi_\sigma &= \bigsqcap_{C \in \sigma \,\cup\, \{\dot{\neg} D \mid D \in \mathsf{sub}(\mathcal{T},\mathcal{R}) \setminus \sigma\}} C \;\; \textit{for each } \sigma \subseteq \mathsf{sub}(\mathcal{T},\mathcal{R}) && (3)\\
\Phi &= \{\Phi_\sigma \mid \sigma \subseteq \mathsf{sub}(\mathcal{T},\mathcal{R})\} && (4)\\
\widehat{\mathsf{sub}}(\mathcal{T},\mathcal{R}) &= \Phi \cup \{\alpha.\beta \mid \alpha \in \{\exists P.\exists P^{\oplus}, \exists P^{\oplus}, \exists P\}, P^{\oplus} \in \mathbf{R}^{+}_{(\mathcal{T},\mathcal{R})}, \beta \in \Phi\} && (5)
\end{aligned}
$$

2.1 Tableaux for $\mathcal{SHIQ}_+$

Tableau structure is introduced to describe a model of a concept w.r.t. a terminology and role hierarchy. Properties in such a tableau definition express semantic constraints resulting directly from the logic constructors in $\mathcal{SHIQ}_+$.

Considering the tableau definition for $\mathcal{SHIQ}$ presented in [11], Definition 3 adopts two additional properties, namely P8 and P9. In particular, P8 imposes a global constraint on a set of individuals of a tableau. This causes the tableaux to lose the local satisfiability property. A tableau has the local satisfiability property if each property of the tableau is related to only one node and its neighbors. This means that, for a graph with a labelling function, checking each node of the graph and its neighbors for each property is sufficient to prove whether this graph is a tableau. The tableau definition for $\mathcal{SHIQ}$ in [11] has the local satisfiability property although $\mathcal{SHIQ}$ includes transitive roles. The propagation of value restrictions on transitive roles by $\forall^{+}$-rule (i.e. the rule for $\forall R.C$ if R is transitive or includes a transitive role) and the absence of number restrictions on transitive roles help to avoid global properties that impose a constraint on an arbitrary set of individuals in a tableau.

Definition 3. *Let $(\mathcal{T},\mathcal{R})$ be a $\mathcal{SHIQ}_+$ knowledge base. A tableau T for a concept D w.r.t $(\mathcal{T},\mathcal{R})$ is defined to be a triplet $(\mathbf{S},\mathcal{L},\mathcal{E})$ such that $\mathbf{S}$ is a set of individuals, $\mathcal{L}$: $\mathbf{S} \rightarrow 2^{\mathsf{sub}(\mathcal{T},\mathcal{R}) \cup \widehat{\mathsf{sub}}(\mathcal{T},\mathcal{R})}$ and $\mathcal{E}$: $\mathbf{R}_{(\mathcal{T},\mathcal{R})} \rightarrow 2^{\mathbf{S}\times\mathbf{S}}$, and there is some individual $s \in \mathbf{S}$ such that $D \in \mathcal{L}(s)$. For all $s \in \mathbf{S}$, $C, C_1, C_2 \in \mathsf{sub}(\mathcal{T},\mathcal{R}) \cup \widehat{\mathsf{sub}}(\mathcal{T},\mathcal{R})$, $R, S \in \mathbf{R}_{(\mathcal{T},\mathcal{R})}$ and $Q^{\oplus} \in \mathbf{R}^{+}_{(\mathcal{T},\mathcal{R})}$, T satisfies the following properties:*

P1 *If* $C_1 \sqsubseteq C_2 \in \mathcal{T}$ *then* $\mathsf{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}(s)$,
P2 *If* $C \in \mathcal{L}(s)$ *then* $\dot{\neg} C \notin \mathcal{L}(s)$,
P3 *If* $C_1 \sqcap C_2 \in \mathcal{L}(s)$ *then* $C_1 \in \mathcal{L}(s)$ *and* $C_2 \in \mathcal{L}(s)$,
P4 *If* $C_1 \sqcup C_2 \in \mathcal{L}(s)$ *then* $C_1 \in \mathcal{L}(s)$ *or* $C_2 \in \mathcal{L}(s)$,
P5 *If* $\forall S.C \in \mathcal{L}(s)$ *and* $\langle s, t\rangle \in \mathcal{E}(S)$ *then* $C \in \mathcal{L}(t)$,
P6 *If* $\forall S.C \in \mathcal{L}(s)$, $Q^{\oplus} \underline{\boxast} S$ *and* $\langle s, t\rangle \in \mathcal{E}(Q)$ *then* $\forall Q^{\oplus}.C \in \mathcal{L}(t)$,
P7 *If* $\exists P.C \in \mathcal{L}(s)$ *with* $P \in \mathbf{R}_{(\mathcal{T},\mathcal{R})} \setminus \mathbf{R}^{+}_{(\mathcal{T},\mathcal{R})}$ *then there is some* $t \in \mathbf{S}$ *such that* $\langle s, t\rangle \in \mathcal{E}(P)$ *and* $C \in \mathcal{L}(t)$,
P8 *If* $\exists Q^{\oplus}.C \in \mathcal{L}(s)$ *then* $(\exists Q.C \sqcup \exists Q.\exists Q^{\oplus}.C) \in \mathcal{L}(s)$, *and there are* $s_1, \cdots, s_n \in \mathbf{S}$ *such that* $\exists Q.C \in \mathcal{L}(s_0) \cup \mathcal{L}(s_{n-1})$ *and* $\langle s_i, s_{i+1}\rangle \in \mathcal{E}(Q)$ *with* $0 \leq i < n$, $s_0 = s$ *and* $\exists Q^{\oplus}.C \in \mathcal{L}(s_j)$ *for all* $0 \leq j < n$.
P9 *If* $\langle s, t\rangle \in \mathcal{E}(Q^{\oplus})$ *then* $\exists Q^{\oplus}.\Phi_\sigma \in \mathcal{L}(s)$ *with* $\sigma = \mathcal{L}(t) \cap \mathsf{sub}(\mathcal{T},\mathcal{R})$ *and*
$$\Phi_\sigma = \bigsqcap_{C \in \sigma \,\cup\, \{\dot{\neg} D \mid D \in \mathsf{sub}(\mathcal{T},\mathcal{R}) \setminus \sigma\}} C,$$
P10 $\langle s, t\rangle \in \mathcal{E}(R)$ *iff* $\langle t, s\rangle \in \mathcal{E}(R^{\ominus})$,
P11 *If* $\langle s, t\rangle \in \mathcal{E}(R)$ *and* $R \underline{\boxast} S$ *then* $\langle s, t\rangle \in \mathcal{E}(S)$,
P12 *If* $(\leq nS.C) \in \mathcal{L}(s)$ *then* $\mathsf{card}\{S^T(s, C)\} \leq n$,
P13 *If* $(\geq nS.C) \in \mathcal{L}(s)$ *then* $\mathsf{card}\{S^T(s, C)\} \geq n$,
P14 *If* $(\leq nS.C) \in \mathcal{L}(s)$ *and* $\langle s, t\rangle \in \mathcal{E}(S)$ *then* $C \in \mathcal{L}(t)$ *or* $\dot{\neg} C \in \mathcal{L}(t)$ *where* $S^T(s, C) := \{t \in \mathbf{S} \mid \langle s, t\rangle \in \mathcal{E}(S) \wedge C \in \mathcal{L}(t)\}$,

Notice that all properties in Definition 3, particularly, the properties P8 and P9 ensure that the label of nodes is a subset of $\mathsf{sub}(\mathcal{T},\mathcal{R}) \cup \widehat{\mathsf{sub}}(\mathcal{T},\mathcal{R})$. In the remainder of the sections, we formulate and prove a lemma that affirms that a tableau represents exactly a model for the concept. P8 in Definition 3 expresses not only the semantic constraint imposed by the transitive closure of roles occurring in concepts such as $\exists Q^{\oplus}.C$ (i.e. a path including nodes are connected by edges containing Q and the label of the last node contains C) but also the non-determinism of transitive closure of roles (i.e. the term $\exists Q.C$ may be chosen at any node of such a path to satisfy $\exists Q^{\oplus}.C$). Additionally, P8 and P9 in Definition 3 enable to satisfy each transitive closure $Q^{\oplus}$ occurring in the label of an edge $\langle s, t\rangle$ with simple role Q. In fact, P9 makes Φ_σ belong to the label of a node t' and s connected to t' by edges containing Q due to P8. The definition of Φ_σ allows t' to be combined with t without causing contradiction. Moreover, this combination does not violate number restrictions since Q is simple. For this reason, the new definition for simple roles presented in Definition 1 is crucial to decidability of $\mathcal{SHIQ}_+$.

In addition, P8 and P9 defined in this way do not require explicitly cycles to satisfy role inclusion axioms such as $R \sqsubseteq Q^{\oplus}$. This makes possible design of tableaux-based algorithm for $\mathcal{SHIQ}_+$ which aims to build tree-like structure i.e. no cycle is explicitly required to be embedded within this structure. The following lemma affirms that a tableau represents exactly a model for the concept.

Lemma 1. *Let* $(\mathcal{T},\mathcal{R})$ *be a* $\mathcal{SHIQ}_+$ *knowledge base. Let* D *be a* $\mathcal{SHIQ}_+$ *concept.* D *is satisfiable w.r.t.* $(\mathcal{T},\mathcal{R})$ *iff there is a tableau for* D *w.r.t.* $(\mathcal{T},\mathcal{R})$.

Proof. • "If-direction". Let $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ be a tableau for $(\mathcal{T}, \mathcal{R})$. A model $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ can be defined as follows:

$\Delta^{\mathcal{I}} = \mathbf{S}$,

$A^{\mathcal{I}} = \{s \mid A \in \mathcal{L}(s)$ for all concept name A in $\mathcal{T}\}$,

$S^{\mathcal{I}} = \mathcal{E}(S) \cup \mathcal{E}'(S) \cup \bigcup_{Q^{\oplus}\underline{\boxast}S \in \mathcal{R}^+} (\mathcal{E}(Q) \cup \mathcal{E}'(Q))^+$ for all role name S in $\mathcal{T}$ and $\mathcal{R}$

where

$\mathcal{E}'(R) = \bigcup_{\mathsf{Cyc}(Q), Q\underline{\boxast}R \in \mathcal{R}^+} \{\langle s, t\rangle \mid \langle s', t'\rangle \in \mathcal{E}(Q), \mathcal{L}'(s) = \mathcal{L}'(s'), \mathcal{L}'(t) = \mathcal{L}'(t')\}$

with $\mathcal{L}'(w) = \mathcal{L}(w) \cap \mathsf{sub}(\mathcal{T}, \mathcal{R})$

Due to $R\underline{\boxast}S \in \mathcal{R}^+$ iff $R^{\ominus}\underline{\boxast}S^{\ominus} \in \mathcal{R}^+$, and $\mathsf{Cyc}(Q)$ iff $\mathsf{Cyc}(Q^{\ominus})$ and the definition of $\mathcal{E}'(S)$, we have :

$$\mathcal{E}'(S^{\ominus}) = \{\langle t, s\rangle \mid \langle s, t\rangle \in \mathcal{E}'(S)\} \tag{6}$$

Due to $\mathsf{P11}$ and the definition of $\mathcal{E}'(S)$, we have :

$$R\underline{\boxast}S \in \mathcal{R}^+ \Rightarrow \mathcal{E}'(R) \subseteq \mathcal{E}'(S) \tag{7}$$

Due to Definition 1, it follows :

$$(S^+)^{\mathcal{I}} = \bigcup_{n>0} \{(\langle s_0, s_1\rangle, \cdots, \langle s_{n-1}, s_n\rangle) \mid \langle s_i, s_{i+1}\rangle \in S^{\mathcal{I}}, 0 \leq i \leq n-1\} \tag{8}$$

To show that $\mathcal{I}$ is a model of $(\mathcal{T}, \mathcal{R})$, we have to show:

1. $\mathcal{I} \models \mathcal{R}$. Assume $R \sqsubseteq S \in \mathcal{R}$. We have to prove $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$. Since $(S^-)^+ \equiv (S^+)^-$, $R \sqsubseteq S$ iff $R^- \sqsubseteq S^-$, and $R \sqsubseteq S^-$ iff $R^- \sqsubseteq S$, it suffices to consider the following unnested cases :
   (a) $R = P^-$ and $S \in \mathbf{R}$. Let $\langle s, t\rangle \in (P^-)^{\mathcal{I}}$. This implies that $\langle t, s\rangle \in P^{\mathcal{I}}$. Due to the definition of $P^{\mathcal{I}}$, we consider the following cases:
   (i) Assume that $\langle t, s\rangle \in \mathcal{E}(P)$. This implies that $\langle s, t\rangle \in \mathcal{E}(P^-)$, and thus $\langle t, s\rangle \in \mathcal{E}(R)$ due to $\mathsf{P10}$ and $\mathsf{P11}$. It follows that $\langle t, s\rangle \in R^{\mathcal{I}}$ due to the definition of $R^{\mathcal{I}}$.
   (ii) Assume that $\langle t, s\rangle \in \mathcal{E}'(P)$ i.e. there are individuals s', t' and some role Q such that $\mathcal{L}'(s) = \mathcal{L}'(s')$, $\mathcal{L}'(t) = \mathcal{L}'(t')$, $\langle t', s'\rangle \in \mathcal{E}(Q)$ with $\mathsf{Cyc}(Q)$ and $Q\underline{\boxast}P \in \mathcal{R}^+$. Due to $\mathsf{P10}$ and $\mathsf{P11}$, this implies that $\langle s', t'\rangle \in \mathcal{E}(Q^{\ominus})$, $\mathsf{Cyc}(Q^{\ominus})$ and $Q^{\ominus}\underline{\boxast}S \in \mathcal{R}^+$. From the definition of $S^{\mathcal{I}}$ it follows that $\langle s, t\rangle \in S^{\mathcal{I}}$.
   (iii) Assume that there are $\langle t, s_1\rangle, \cdots, \langle s_n, s\rangle \in \mathcal{E}(Q) \cup \mathcal{E}'(Q)$ with $Q^{\oplus}\underline{\boxast}P \in \mathcal{R}^+$. This implies that $\langle s, s_n\rangle, \cdots, \langle s_1, t\rangle \in \mathcal{E}(Q^{\ominus}) \cup \mathcal{E}'(Q^{\ominus})$ due to $\mathsf{P10}$ and (6). Moreover, from $Q^{\oplus}\underline{\boxast}P \in \mathcal{R}^+$ it follows that $(Q^{\ominus})^{\oplus}\underline{\boxast}P^{\ominus} \in \mathcal{R}^+$, and thus $(Q^{\ominus})^{\oplus}\underline{\boxast}R \in \mathcal{R}^+$. Due to the definition of $R^{\mathcal{I}}$, we have $\langle s, t\rangle \in R^{\mathcal{I}}$.
   (b) $R = P^+$ and $S \in \mathbf{R}$. Let $\langle s, t\rangle \in (P^+)^{\mathcal{I}}$. From (8), there are $\langle s, s_1\rangle, \cdots, \langle s_n, t\rangle \in P^{\mathcal{I}}$. Due to the definition of $P^{\mathcal{I}}$, we consider the following cases:
   Assume that $\langle s_i, s_{i+1}\rangle \in \mathcal{E}(P) \cup \mathcal{E}'(P)$.

Assume that there are $\langle s_i, w_1\rangle, \cdots, \langle w_m, s_{i+1}\rangle \in \mathcal{E}(P') \cup \mathcal{E}'(P')$ where $P'^{+} \underline{\boxast} P \in \mathcal{R}^{+}$ with some $i \in \{0, n+1\}, s_0 = s, s_{n+1} = t$. Since $P'^{+} \underline{\boxast} P \in \mathcal{R}^{+}$ implies $P' \underline{\boxast} P \in \mathcal{R}^{+}$, P11 and (7), it holds that $\langle s_i, w_1\rangle, \cdots, \langle w_m, s_{i+1}\rangle \in \mathcal{E}(P) \cup \mathcal{E}'(P)$. This implies that

$$\exists u_1, \cdots, u_k : \langle s, u_1\rangle, \cdots, \langle u_k, t\rangle \in \mathcal{E}(P) \cup \mathcal{E}'(P) \qquad (9)$$

According to the definition of $S^{\mathcal{I}}$ with $P^{+} \underline{\boxast} S \in \mathcal{R}^{+}$ and (10), we have $\langle s, t\rangle \in S^{\mathcal{I}}$.

(c) $R \in \mathbf{R}$ and $S = Q^{+}$. Let $\langle s, t\rangle \in R^{\mathcal{I}}$. According to the definition of $R^{\mathcal{I}}$, we consider the following cases:
(i) Assume that $\langle s, t\rangle \in \mathcal{E}(R)$ and $\langle s, t\rangle \notin \mathcal{E}(Q)$ (if $\langle s, t\rangle \in \mathcal{E}(Q)$ then $\langle s, t\rangle \in Q^{\mathcal{I}} \subseteq (Q^{+})^{\mathcal{I}}$). By P11, we have $\langle s, t\rangle \in \mathcal{E}(Q^{+})$. Moreover, due to P7, P8 and P9 there are $\langle s, s_1\rangle, \cdots, \langle s_{n-1}, s_n\rangle \in \mathcal{E}(Q)$ with $\varPhi_\sigma = \bigsqcap_{C \in \sigma \cup \{\dot{\neg} D \mid D \in \mathsf{sub}(\mathcal{T},\mathcal{R}) \setminus \sigma\}} C \in \mathcal{L}(s_n)$ and $\sigma = \mathcal{L}(t) \cap \mathsf{sub}(\mathcal{T},\mathcal{R}) = \mathcal{L}'(t)$. Due to P3 we have $\mathcal{L}'(t) \subseteq \mathcal{L}'(s_n)$. Let $D \in \mathcal{L}(s_n) \cap \mathsf{sub}(\mathcal{T},\mathcal{R})$. If $D \notin \mathcal{L}(t) \cap \mathsf{sub}(\mathcal{T},\mathcal{R})$ then, by the definition of $\varPhi_\sigma$, $\dot{\neg} D \in \mathcal{L}(t) \cap \mathsf{sub}(\mathcal{T},\mathcal{R})$. By P2, this is not possible, and thus $\mathcal{L}'(t) = \mathcal{L}'(s_n)$. Due to the definition of $Q^{\mathcal{I}}$ we have

$$\langle s_{n-1}, t\rangle \in Q^{\mathcal{I}} \textit{ and } \langle s, s_1\rangle, \cdots, \langle s_{n-2}, s_{n-1}\rangle \in Q^{\mathcal{I}} \qquad (10)$$

(ii) Assume that $\langle s, t\rangle \in \mathcal{E}'(R)$. According to the definition of $\mathcal{E}'(R)$ there are individuals s', t' and some role P such that $\mathcal{L}'(s) = \mathcal{L}'(s')$, $\mathcal{L}'(t) = \mathcal{L}'(t')$, $\langle t', s'\rangle \in \mathcal{E}(P)$ with $\mathsf{Cyc}(P)$ and $P \underline{\boxast} R \in \mathcal{R}^{+}$. Due to P11 we have $\langle s', t'\rangle \in \mathcal{E}(Q^{+})$. Assume that $\langle s', t'\rangle \notin \mathcal{E}(Q)$ (if $\langle s', t'\rangle \in \mathcal{E}(Q)$ then $\langle s, t\rangle \in Q^{\mathcal{I}} \subseteq (Q^{+})^{\mathcal{I}}$). Due to P7, P8 and P9 there are $\langle s', s_1\rangle, \cdots, \langle s_{n-1}, s_n\rangle \in \mathcal{E}(Q)$ with $\varPhi_\sigma \in \mathcal{L}(s_n)$ and $\sigma = \mathcal{L}(t') \cap \mathsf{sub}(\mathcal{T},\mathcal{R}) = \mathcal{L}'(t')$. Since $\mathcal{L}'(s) = \mathcal{L}'(s')$, $\mathcal{L}'(t) = \mathcal{L}'(t') = \mathcal{L}'(s_n)$, $\mathsf{Cyc}(Q)$, we have

$$\langle s, s_1\rangle, \langle s_{n-1}, t\rangle \in Q^{\mathcal{I}} \textit{ and } \langle s_1, s_2\rangle, \cdots, \langle s_{n-2}, s_{n-1}\rangle \in Q^{\mathcal{I}} \qquad (11)$$

due to the definition of $Q^{\mathcal{I}}$.
(iii) Assume that there are $\langle s, s_1\rangle, \cdots, \langle s_n, t\rangle \in \mathcal{E}(P) \cup \mathcal{E}'(P)$ with $P^{+} \underline{\boxast} R \in \mathcal{R}^{+}$. Since $P^{+} \underline{\boxast} R \in \mathcal{R}^{+}$ implies $P \underline{\boxast} R \in \mathcal{R}^{+}$, P11 and (7), it holds that there are $\langle s, u_1\rangle, \cdots, \langle u_m, t\rangle \in \mathcal{E}(R) \cup \mathcal{E}'(R)$. From (10) and (11) we obtain

$$\exists v_1, \cdots, v_k : \langle s, v_1\rangle, \cdots, \langle v_k, t\rangle \in Q^{\mathcal{I}} \qquad (12)$$

(d) $R = P^{+}$ and $S = Q^{+}$. Due to (10), (11) and (12).

2. $\mathcal{I} \models \mathcal{T}$ i.e. if $C \sqsubseteq D \in \mathcal{T}$ then $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
3. $D^{\mathcal{I}} \neq \emptyset$.

The items 2 and 3 are proved if we can show that $C \in \mathcal{L}(s)$ implies $s \in C^{\mathcal{I}}$ for all $s \in \mathbf{S}$ (**). In fact, due to P1 it follows that $\mathsf{nnf}(\neg C \sqcup D) \in \mathcal{L}(s)$ for all $s \in \mathbf{S}$ and $C \sqsubseteq D \in \mathcal{T}$. Due to (**) and Definition 2, we have $s \in (\mathsf{nnf}(\neg C \sqcup D))^{\mathcal{I}} = (\neg C \sqcup D)^{\mathcal{I}} = (\neg C)^{\mathcal{I}} \cup D^{\mathcal{I}}$ for all $s \in \mathbf{S}$ and $C \sqsubseteq D \in \mathcal{T}$. This implies that if $s \in C^{\mathcal{I}}$ then $s \in D^{\mathcal{I}}$. Therefore, the item 2 is shown.

Moreover, since T is a tableau for D and thus, there exists $s \in \mathbf{S}$ such that $D \in \mathcal{L}(s)$. Due to (**) it follows $s \in D^{\mathcal{I}} \neq \emptyset$. Therefore, the item 3 is shown.
We now prove (**) by induction on the length of a concept C, denoted $len(C)$ where C in NNF, is defined as follows:

$len(A) := len(\neg A) := 0$
$len(C_1 \sqcap C_2) := len(C_1 \sqcup C_2) := 1 + len(C_1) + len(C_2)$
$len(\forall R.C) := len(\exists R.C) := 1 + len(C)$

Two basic cases are $C = A$ or $C = \neg A$. If $A \in \mathcal{L}(s)$ then, by the definition of $\mathcal{I}$, $s \in A^{\mathcal{I}}$. If $\neg A \in \mathcal{L}(s)$ then, by P2, $A \notin \mathcal{L}(s)$ and thus $s \notin A^{\mathcal{I}}$. For the inductive step, we have to distinguish several cases:

- $C = C_1 \sqcap C_2$. P3 and $C \in \mathcal{L}(s)$ imply $C_1, C_2 \in \mathcal{L}(s)$. By induction, we have $s \in C_1^{\mathcal{I}}$ and $s \in C_2^{\mathcal{I}}$. Since $\mathcal{I}$ is an interpretation (Definition 2) hence $s \in (C_1 \sqcap C_2)^{\mathcal{I}}$.
- $C = C_1 \sqcup C_2$. The same argument.
- $C = \exists S.E$ with $S \in \mathbf{R}_{(\mathcal{T},\mathcal{R})} \setminus \mathbf{R}^+_{(\mathcal{T},\mathcal{R})}$. According to P7, there is some $t \in \mathbf{S}$ such that $E \in \mathcal{L}(t)$ and $\langle s, t\rangle \in \mathcal{E}(S)$. By induction, we have $t \in E^{\mathcal{I}}$. From the definition of $S^{\mathcal{I}}$ it follows that $\langle s, t\rangle \in S^{\mathcal{I}}$ and thus $s \in C^{\mathcal{I}}$.
- $C = \exists Q^{\oplus}.E$. According to P8, there are $s_1, \cdots, s_n$ such that $\langle s, s_1\rangle, \cdots, \langle s_{n-1}, s_n\rangle \in \mathcal{E}(Q)$ and $E \in \mathcal{L}(s_n)$. By induction, we have $s_n \in E^{\mathcal{I}}$. Moreover, from the definition of $Q^{\mathcal{I}}$ it follows that $\langle s, s_1\rangle, \cdots, \langle s_{n-1}, s_n\rangle \in Q^{\mathcal{I}}$ and thus $s \in C^{\mathcal{I}}$.
- $C = \forall S.E$. Let $t \in \mathbf{S}$ be an individual such that $\langle s, t\rangle \in S^{\mathcal{I}}$. According to the definition of $S^{\mathcal{I}}$, we consider the following cases:
  - $\langle s, t\rangle \in \mathcal{E}(S)$. According to P5 it follows that $E \in \mathcal{L}(t)$. By induction, we have $t \in E^{\mathcal{I}}$ and thus $s \in C^{\mathcal{I}}$.
  - $\langle s, t\rangle \in \mathcal{E}'(S)$. This implies that there are indivuduals s', t' and some role P such that $\mathcal{L}'(s) = \mathcal{L}'(s')$, $\mathcal{L}'(t) = \mathcal{L}'(t')$, $\langle t', s'\rangle \in \mathcal{E}(P)$ with $\mathsf{Cyc}(P)$ and $P \underline{\circledast} S \in \mathcal{R}^+$. We have $\forall S.E \in \mathsf{sub}(\mathcal{T},\mathcal{R})$ and thus $\forall S.E \in \mathcal{L}'(s')$. Due to P11 and $P \underline{\circledast} S \in \mathcal{R}^+$ we have $\langle t', s'\rangle \in \mathcal{E}(S)$. From P5 it follows $E \in \mathcal{L}'(t') = \mathcal{L}'(t) \subseteq \mathcal{L}(t)$. By induction, we have $t \in E^{\mathcal{I}}$ and thus $s \in C^{\mathcal{I}}$.
  - $\langle s, s_1\rangle, \cdots, \langle s_n, t\rangle \in \mathcal{E}(Q) \cup \mathcal{E}'(Q)$ with $Q^{\oplus} \underline{\circledast} S \in \mathcal{R}^+$. By the definition of $Q^{\mathcal{I}}$, we have $\langle s, s_1\rangle, \cdots, \langle s_n, s_\rangle \in Q^{\mathcal{I}}$. Moreover, according to P6 (with $Q^{\oplus} \underline{\circledast} Q^{\oplus} \in \mathcal{R}^+$ and the same argument above if $\langle s_i, s_{i+1}\rangle \in \mathcal{E}'(Q)$), it follows that $\forall Q^{\oplus}.E \in \mathcal{L}(s_i)$ for all $i \in \{1, \cdots n\}$. Due to $Q \underline{\circledast} Q^{\oplus} \in \mathcal{R}^+$ and P11 it follows that $\langle s_n, t\rangle \in \mathcal{E}(Q^{\oplus})$. This implies that $E \in \mathcal{L}(t)$ due to P5. By induction, we have $t \in E^{\mathcal{I}}$ and hence $s \in C^{\mathcal{I}}$.
- $C = (\geq nS.E)$ where S is simple. We have $\#S^T(s, E) \geq n$. This means that there are $s_1, \cdots, s_n \in \mathbf{S}$ such that $\langle s, s_i\rangle, \cdots, \langle s, s_i\rangle \in \mathcal{E}(S)$, $E \in \mathcal{L}(s_i)$ with $s_i \neq s_j$ for all $i \neq j$. By induction, we have $s_i \in E^{\mathcal{I}}$, and $\langle s, s_i\rangle \in S^{\mathcal{I}}$ since $\mathcal{E}(S) \subseteq S^{\mathcal{I}}$. Thus, $s \in (\geq nS.E)^{\mathcal{I}}$.
- $C = (\leq nS.E)$ where S is simple. Since S is simple, according to the definition of $\mathcal{I}$, we have $\mathcal{E}'(S) = \emptyset$ and $\bigcup_{Q^{\oplus} \underline{\circledast} S \in \mathcal{R}^+} (\mathcal{E}(Q) \cup \mathcal{E}'(Q))^+ = \emptyset$. Thus, $S^{\mathcal{I}} = \mathcal{E}(S)$. Moreover, according to P12, $\#S^T(s, C) \leq n$. We try to show $\#S^{\mathcal{I}}(s, E) \leq \#S^T(s, E)$. By absurdity, assume that there is some $t \in \mathbf{S}$ such that $t \in E^{\mathcal{I}}$, $\langle s, t\rangle \in S^{\mathcal{I}}$ but $E \notin \mathcal{L}(t)$ (since $\mathcal{E}(S) = S^{\mathcal{I}}$). According to P14, we have $\dot{\neg} E \in \mathcal{L}(t)$. By induction, we have $t \in (\dot{\neg} E)^{\mathcal{I}}$ which contradicts $t \in E^{\mathcal{I}}$.

• "Only-If-direction". Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $(\mathcal{T}, \mathcal{R})$.
Let $\Phi_1(s) = \{C \in \mathsf{sub}(\mathcal{T}, \mathcal{R}) \mid s \in C^{\mathcal{I}}\}$. First, we show that

$$\textit{For each } s \in \Delta^{\mathcal{I}}\textit{, it holds that } s \in \Big(\bigsqcap_{C \in \Phi_1(s) \,\cup\, \{\dot{\neg} D \mid D \in \mathsf{sub}(\mathcal{T},\mathcal{R}) \setminus \Phi_1(s)\}} C\Big)^{\mathcal{I}} \quad (13)$$

In fact, for each $D \in \mathsf{sub}(\mathcal{T}, \mathcal{R})$ and for each $s \in \Delta^{\mathcal{I}}$ it holds that $s \in (\neg D \sqcup D)^{\mathcal{I}} = \Delta^{\mathcal{I}}$. Moreover, if $D \notin \{C \in \mathsf{sub}(\mathcal{T}, \mathcal{R}) \mid s \in C^{\mathcal{I}}\}$ then $s \notin D^{\mathcal{I}}$ and thus $s \in (\dot{\neg} D)^{\mathcal{I}} = (\neg D)^{\mathcal{I}}$.
A tableau $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ for $(\mathcal{T}, \mathcal{R})$ can be defined as follows:
$\mathbf{S} \quad = \Delta^{\mathcal{I}}$,
$\mathcal{E}(R) = R^{\mathcal{I}}$ for all role R occurring in $\mathcal{T}$ and $\mathcal{R}$,
$\mathcal{L}(s) = \{E \in \mathsf{sub}(\mathcal{T}, \mathcal{R}) \cup \{\bigsqcap_{C \in \Phi_1(s) \,\cup\, \{\dot{\neg} D \mid D \in \mathsf{sub}(\mathcal{T},\mathcal{R}) \setminus \Phi_1(s)\}} C\} \mid s \in E^{\mathcal{I}}\}$

We now show that T is a tableau of $(\mathcal{T}, \mathcal{R})$.

– P1, P2, P3, P4. Obvious.
– P5. Let $\forall S.C \in \mathcal{L}(s)$ and $\langle s, t\rangle \in \mathcal{E}(S)$. According to the definition of T, $\langle s, t\rangle \in \mathcal{E}(S) = S^{\mathcal{I}}$. Since $\mathcal{I}$ is a model, we have $t \in C^{\mathcal{I}}$. Due to the definition of T, we have $C \in \mathcal{L}(t)$.
– P6. Let $\forall S.C \in \mathcal{L}(s)$ with $Q^{\oplus} \underline{\circledast} S$ and $\langle s, t\rangle \in \mathcal{E}(Q)$. According to the definition of T, $\langle s, t\rangle \in \mathcal{E}(Q) = Q^{\mathcal{I}}$. We have to show that $\forall Q^{\oplus}.C \in \mathcal{L}(t)$. Assume that there are $\langle t, t_1\rangle, \cdots, \langle t_{n-1}, t_n\rangle \in Q^{\mathcal{I}}$. Due to that $Q^{\oplus} \underline{\circledast} S$ and $\mathcal{I}$ is a model, we have $\langle s, t_n\rangle \in S^{\mathcal{I}}$, and thus $t_n \in C^{\mathcal{I}}$. This implies that $t \in (\forall Q^{\oplus}.C)^{\mathcal{I}}$. By the definition of T, we have $\forall Q^{\oplus}.C \in \mathcal{L}(t)$.
– P7. Let $\exists S.C \in \mathcal{L}(s)$. There is some t such that $\langle s, t\rangle \in S^{\mathcal{I}}$ and $t \in C^{\mathcal{I}}$. According to the definition of T, $\langle s, t\rangle \in \mathcal{E}(S) = S^{\mathcal{I}}$ and $C \in \mathcal{L}(t)$.
– P8. Let $\exists P^{\oplus}.C \in \mathcal{L}(s)$. There are $\langle s, s_1\rangle, \cdots, \langle s_{n-1}, s_n\rangle \in P^{\mathcal{I}}$, $s_n \in C^{\mathcal{I}}$ and $\exists P^{\oplus}.C \in \mathcal{L}(s_i)$ with $0 \leq i < n$, $s_0 = s$. According to the definition of T, we have $\langle s_i, s_{i+1}\rangle \in \mathcal{E}(P)$ such that $C \in \mathcal{L}(s_n)$, $\exists Q.C \in \mathcal{L}(s_{n-1})$, $(\exists P^{\oplus}.C) \in \mathcal{L}(s_i)$ with $0 \leq i < n$. We show that $\exists P.C \in \mathcal{L}(s)$ or $\exists P.\exists P^{\oplus}.C \in \mathcal{L}(s)$.
If $n = 1$ then $C \in \mathcal{L}(s_1)$ and $s_1 \in C^{\mathcal{I}}$. This implies $s \in (\exists P.C)^{\mathcal{I}}$. Due to the definition of T, we have $\exists P.C \in \mathcal{L}(s)$.
Assume that $n > 1$. We have $s_1 \in (\exists P^{\oplus}.C)^{\mathcal{I}}$. This implies that $s \in (\exists P.\exists P^{\oplus}.C)^{\mathcal{I}}$ due to $\langle s, s_1\rangle \in P^{\mathcal{I}}$. From the definition of T, it follows that $\exists P.\exists P^{\oplus}.C \in \mathcal{L}(s)$.
– P9. Let $\langle s, t\rangle \in \mathcal{E}(Q^{\oplus})$. This implies that there are $\langle s, s_1\rangle, \cdots, \langle s_n, t\rangle \in Q^{\mathcal{I}}$. From the definition of T, we have $\langle s, s_1\rangle, \cdots, \langle s_n, t\rangle \in \mathcal{E}(Q)$. Due to the definition of T and (13), we have $t \in (\Phi_\sigma)^{\mathcal{I}}$ with $\sigma = \mathcal{L}(t) \cap \mathsf{sub}(\mathcal{T}, \mathcal{R})$. This implies $s \in (\exists Q^{\oplus}.\Phi_\sigma)^{\mathcal{I}}$. From the definition of T, we have $(\exists Q^{\oplus}.\Phi_\sigma) \in \mathcal{L}(s)$.
– P10-P14. Obvious.

3 A tableaux-based decision procedure for $\mathcal{SHIQ}_+$

As mentioned, a tableau for a concept represents a model that is possibly infinite. However, the goal of a tableaux-based algorithm is to find a finite structure that has to imply a tableau. Conversely, the existence of a tableau can guide us to build such a structure. Such a finite structure is introduced in Definition 4, namely, completion tree.

Definition 4. *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHIQ}_+$ knowledge base. Let D be a $\mathcal{SHIQ}_+$ concept. A* completion tree *for D and $(\mathcal{T}, \mathcal{R})$ is a tree $\mathbf{T} = (V, E, \mathcal{L}, x_{\mathbf{T}}, \dot{\neq})$ where*

$*$ *V is a set of nodes containing a root node $x_{\mathbf{T}} \in V$. Each node $x \in V$ is labelled with a function $\mathcal{L}$ such that $\mathcal{L}(x) \subseteq \mathsf{sub}(\mathcal{T}, \mathcal{R}) \cup \widehat{\mathsf{sub}}(\mathcal{T}, \mathcal{R})$. In addition, $\dot{\neq}$ is a symetric binary relation over V.*

$*$ *E is a set of edges. Each edge $\langle x, y \rangle \in E$ is labelled with a function $\mathcal{L}$ such that $\mathcal{L}(\langle x, y \rangle) \subseteq \mathbf{R}_{(\mathcal{T}, \mathcal{R})}$.*

$*$ *If $\langle x, y \rangle \in E$ then y is called a successor of x, denoted by $y \in \mathsf{succ}^1(x)$, or x is called the predecessor of y, denoted by $x = \mathsf{pred}^1(y)$. In this case, we say that x is a neighbor of y or y is a neighbor of x. If $z \in \mathsf{succ}^n(x)$ (resp. $z = \mathsf{pred}^n(x)$) and y is a successor of z (resp. y is the predecessor of z) then $y \in \mathsf{succ}^{(n+1)}(x)$ (resp. $y = \mathsf{pred}^{(n+1)}(x)$) for all $n \geq 0$ where $\mathsf{succ}^0(x) = \{x\}$ and $\mathsf{pred}^0(x) = x$.*

$*$ *A node y is called a R-successor of x, denoted by $y \in \mathsf{succ}^1_R(x)$ (resp. y is called the R-predecessor of x, denoted by $y = \mathsf{pred}^1_R(x)$) if there is some role R' such that $R' \in \mathcal{L}(\langle x, y \rangle)$ (resp. $R' \in \mathcal{L}(\langle y, x \rangle)$) and $R' \underline{\circledast} R$. A node y is called a R-neighbor of x if y is either a R-successor or R-predecessor of x. If z is a R-successor of y (resp. z is the R-predecessor of y) and $y \in \mathsf{succ}^n_R(x)$ (resp. $y = \mathsf{pred}^n_R(x)$) then $z \in \mathsf{succ}^{(n+1)}_R(x)$ (resp. $z = \mathsf{pred}^{(n+1)}_R(x)$) for $n \geq 0$ with $\mathsf{succ}^0_R(x) = \{x\}$ and $x = \mathsf{pred}^0_R(x)$.*

$*$ *For a node x and a role S, we define the set $S^{\mathbf{T}}(x, C)$ of x's S-neighbors as follows:*

$$S^{\mathbf{T}}(x, C) = \{y \in V \mid y \text{ is a } S\text{-neighbor of } x \text{ and } C \in \mathcal{L}(x)\}$$

$*$ *A node x is called* blocked *by y, denoted by $y = \mathsf{b}(x)$, if there are numbers $n, m > 0$ and nodes x', y, y' such that*

1. *$x_{\mathbf{T}} = \mathsf{pred}^n(y)$, $y = \mathsf{pred}^m(x)$, and*
2. *$x' = \mathsf{pred}^1(x)$, $y' = \mathsf{pred}^1(y)$, and*
3. *$\mathcal{L}(x) = \mathcal{L}(y)$, $\mathcal{L}(x') = \mathcal{L}(y')$, and*
4. *$\mathcal{L}(\langle x', x \rangle) = \mathcal{L}(\langle y', y \rangle)$, and*
5. *if there are z, z' such that $z' = \mathsf{pred}^1(z)$, $\mathsf{pred}^i(z') = x_{\mathbf{T}}$, $\mathcal{L}(z) = \mathcal{L}(y)$, $\mathcal{L}(z') = \mathcal{L}(y')$ and $\mathcal{L}(\langle z', z \rangle) = \mathcal{L}(\langle y', y \rangle)$ then $n \leq i$.*

$*$ *We define an extended function $\widehat{\mathsf{succ}}$ from succ over $\mathbf{T}$ as follows:*

- *if x has a successor y (resp. x has a R-successor y) that is not blocked then $y \in \widehat{\mathsf{succ}}^1(x)$ (resp. $y \in \widehat{\mathsf{succ}}^1_R(x)$),*
- *if x has a successor z (resp. x has a R-successor z) that is blocked by $\mathsf{b}(z)$ then $\mathsf{b}'(z) \in \widehat{\mathsf{succ}}^1(x)$ (resp. $\mathsf{b}(z) \in \widehat{\mathsf{succ}}^1_R(x)$).*
- *if $y \in \widehat{\mathsf{succ}}^n_R(x)$ and $z \in \widehat{\mathsf{succ}}^1_R(y)$ then $z \in \widehat{\mathsf{succ}}^{(n+1)}_R(x)$ for $n \geq 0$.*

$*$ *A node z is called a $\exists R^{\oplus}.C$-reachable of x with $\exists R^{\oplus}.C \in \mathcal{L}(x)$ if there are $x_1, \cdots, x_{k+n} \in V$ with $x_{k+n} = z, x_0 = x$ and $k + n \geq 0$ such that $x_i = \mathsf{pred}^i_R(x_0)$, $\exists R^{\oplus}.C \in \mathcal{L}(x_i)$ with $i \in \{0, \cdots, k\}$, and $x_{j+k} \in \widehat{\mathsf{succ}}^j_R(x_k)$, $\exists R^{\oplus}.C \in \mathcal{L}(x_{j+k})$, $\exists R.C \in \mathcal{L}(x_{(k+n)})$ with $j \in \{0, \cdots, n\}$.*

$*$ **Clashes** : *$\mathbf{T}$ is said to contain a* clash *if one of the following conditions holds:*

1. *There is some node $x \in V$ such that $\{A, \dot{\neg}A\} \subseteq \mathcal{L}(x)$ for some concept name $A \in \mathbf{C}$,*
2. *There is some node $x \in V$ with $(\leq nS.C) \in \mathcal{L}(x)$ and there are $(n+1)$ S-neighbors $y_1, \cdots, y_{n+1}$ of x such that $y_i \dot{\neq} y_j$ and $C \in \mathcal{L}(x_i)$ for all $1 \leq i < j \leq (n+1)$,*
3. *There is some node $x \in V$ with $\exists R^{\oplus}.C \in \mathcal{L}(x)$ such that there does not exist any $\exists R^{\oplus}.C$-reachable node y of x,*

The definition of tableaux provides a strategy to design an algorithm that can be described by a set of rules, namely expansion rules. Algorithm 2 builds a completion tree for a $\mathcal{SHIQ}_+$ concept by applying the expansion rules in Figure 2 and 3. The expansion rules in 2 were given in [11]. We introduce two new expansion rules that correspond to P8 and P9 in Definition 3.

Algorithm 2 builds a completion tree for a $\mathcal{SHIQ}_+$ concept by applying the expansion rules in Figure 2 and 3. The expansion rules in Figure 2 were given in [11]. We introduce two new expansion rules that correspond to P8 and P9 in Definition 3.

In comparison with $\mathcal{SHIQ}$, there is a new source of non-determinisms that could augment the complexity of an algorithm for satisfiablity of concepts in $\mathcal{SHIQ}_+$. This source comes from the presence of transitive closure of role in concepts. This means that for each occurrence of a term such as $\exists Q^{\oplus}.C$ in the label of a node of a completion tree we have to check the existence of a sequence of edges such that the label of each edge contains Q and the label of the last node contains C. The process for checking the existence of paths whose length is arbitrary must be translated into a process that works for a finite structure. To do this, we reuse the blocking condition introduced in [11] and introduce a function $\widehat{\mathsf{succ}}(x)$ that returns the set of x's successors in a completion tree. An infinite path over a completion tree can be defined thanks to this function. The $\exists_+$-rule in Figure 3 generates all possible paths. The clash-freeness of the third kind in Definition 4 ensures that a "good" path has to be picked from this set of all possible paths.

The function $\mathsf{checkReachability}^{Q}_{C}(x, d, \mathcal{B})$ depicted in Algorithm 1 represents an algorithm for checking the clash-freeness of the third kind for a completion tree. It returns a $\exists Q^{\oplus}.C$-reachable node of x if there exists one. In this function, the parameter x represents a node of the tree to be checked i.e. there is a term such as $\exists Q^{\oplus}.C \in \mathcal{L}(x)$. The parameter d indicates the direction to search from x. Depending on $d = 1$ or $d = 0$, the algorithm goes up to ancestors of x or goes down to descendants of x respectively. When the algorithm goes down, it never goes up again. The subset $\mathcal{B} \subseteq V$ represents the set of all blocked nodes among the nodes that the algorithm have visited. The function $\mathsf{checkReachability}^{Q}_{C}(x, 1, \emptyset)$ would be called for each non-blocked node x of a completion tree and for each term such as $\exists Q^{\oplus}.C \in \mathcal{L}(x)$.

As shown in Lemma 2, the complexity of Algorithm 1 is bounded by a double exponential function in size of inputs.

Lemma 2 (Termination). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHIQ}_+$ knowledge base. Let D be a $\mathcal{SHIQ}_+$-concept w.r.t. $(\mathcal{T}, \mathcal{R})$. Algorithm 2 terminates.*

Proof. The termination of Algorithm 2 is a consequence of the following claims:

$\sqsubseteq$-rule: if $C \sqsubseteq D \in \mathcal{T}$ and $\mathsf{nnf}(\neg C \sqcup D) \notin \mathcal{L}(x)$
then $\mathcal{L}(x) \longleftarrow \mathcal{L}(x) \cup \{\mathsf{nnf}(\neg C \sqcup D)\}$

$\sqcap$-rule: if $C_1 \sqcap C_2 \in \mathcal{L}(x)$ and $\{C_1, C_2\} \not\subseteq \mathcal{L}(x)$
then $\mathcal{L}(x) \longleftarrow \mathcal{L}(x) \cup \{C_1, C_2\}$

$\sqcup$-rule: if $C_1 \sqcup C_2 \in (x)$ and $\{C_1, C_2\} \cap \mathcal{L}(x) = \emptyset$
then $\mathcal{L}(x) \longleftarrow \mathcal{L}(x) \cup \{C\}$ for some $C \in \{C_1, C_2\}$

$\exists$-rule: if 1. $\exists S.C \in \mathcal{L}(x)$, x is not blocked, and
2. x has no S-neighbour y with $C \in \mathcal{L}(y)$
then create a new node y with $\mathcal{L}(\langle x, y\rangle)$=$\{S\}$ and $\mathcal{L}(y)$=$\{C\}$

$\forall$-rule: if 1. $\forall S.C \in \mathcal{L}(x)$, and
2. there is a S-neighbour y of x such that $C \notin \mathcal{L}(y)$
then $\mathcal{L}(y) \longleftarrow \mathcal{L}(y) \cup \{C\}$

$\forall_+$-rule: if 1. $\forall S.C \in \mathcal{L}(x)$, and
2. there is some Q with $Q^{\oplus} \sqsubseteq^{*} S$, and
3. there is an Q-neighbour y of x such that $\forall Q^{\oplus}.C \notin \mathcal{L}(y)$
then $\mathcal{L}(y) \longleftarrow \mathcal{L}(y) \cup \{\forall Q^{\oplus}.C\}$

ch-rule: if 1. $(\leq n\, S.C) \in \mathcal{L}(x)$, and
2. there is an S-neighbour y of x with $\{C, \dot{\neg} C\} \cap \mathcal{L}(y) = \emptyset$
then $\mathcal{L}(y) \longleftarrow \mathcal{L}(y) \cup \{E\}$ for some $E \in \{C, \dot{\neg} C\}$

$\geq$-rule: if 1. $(\geq n\, S.C) \in \mathcal{L}(x)$ and x is not blocked, and
2. there are no n S-neighbors $y_1, ..., y_n$ such that $C \in \mathcal{L}(y_i)$, and $y_i \neq y_j$ for $1 \leq i < j \leq n$,
then create n new nodes $y_1, ..., y_n$ with $\mathcal{L}(\langle x, y_i\rangle)$=$\{S\}$,
$\mathcal{L}(y_i)$=$\{C\}$, and $y_i \neq y_j$ for $1 \leq i < j \leq n$.

$\leq$-rule: if 1. $(\leq n\, S.C) \in \mathcal{L}(x)$, and
2. $\mathsf{card}\{S^{\mathbf{T}}(x, C)\} > n$ and there are two S-neighbors y, z of x with $C \in \mathcal{L}(y) \cap \mathcal{L}(z)$, y is not an ancestor of z, and not $y \neq z$
then 1. $\mathcal{L}(z) \longleftarrow \mathcal{L}(z) \cup \mathcal{L}(y)$ and $\mathcal{L}(\langle x, y\rangle) \longleftarrow \emptyset$
2. If z is an ancestor of x
then $\mathcal{L}(\langle z, x\rangle) \longleftarrow \mathcal{L}(\langle z, x\rangle) \cup \{R^{\ominus} \mid R \in \mathcal{L}(\langle x, y\rangle)\}$
else $\mathcal{L}(\langle x, z\rangle) \longleftarrow \mathcal{L}(\langle x, z\rangle) \cup \mathcal{L}(\langle x, y\rangle)$
4. Add $u \neq z$ for all u such that $u \neq y$

Fig. 2. Expansion rules for $\mathcal{SHIQ}$ presented in [11]

$\exists_+$-rule: if $\exists S^{\oplus}.C \in \mathcal{L}(x)$ and $(\exists S.C \sqcup \exists S.\exists S^{\oplus}.C) \notin \mathcal{L}(x)$
then $\mathcal{L}(x) \longleftarrow \mathcal{L}(x) \cup \{\exists S.C \sqcup \exists S.\exists S^{\oplus}.C\}$

$\oplus$-rule: if x has a $P^{\oplus}$-neighbor y and $\exists P^{\oplus}.\Phi_\sigma \notin \mathcal{L}(x)$ with $\sigma = \mathcal{L}(y) \cap \mathsf{sub}(\mathcal{T}, \mathcal{R})$
then $\mathcal{L}(x) = \mathcal{L}(x) \cup \{\exists P^{\oplus}.\Phi_\sigma\}$

Fig. 3. New expansion rules for $\mathcal{SHIQ}_+$

```
checkReachability_C^Q(x, d, B)
if ∃Q.C ∈ L(x) then
    return true;
if d = 1 then
    if there is pred_Q^1(x) with ∃Q^⊕.C ∈ L(pred_Q^1(x)) then
        checkReachability_C^Q(pred_Q^1(x), 1, B) ;
foreach x' ∈ succ_Q^1(x) such that ∃Q^⊕.C ∈ L(x') do
    if ∃Q.C ∈ L(x') then
        return true;
    if x' is not blocked then
        checkReachability_C^Q(x', 0, B) ;
    else
        if x' ∉ B then
            B = B ∪ {x'};
            checkReachability_C^Q(b(x'), 0, B) ;
return false;
```

Algorithm 1: $\mathsf{checkReachability}_C^Q(x, d, \mathcal{B})$ for checking the existence of a $\exists Q^{\oplus}.C$-reachable node of $x \in V$ where $d \in \{1, 0\}$, $\mathcal{B} \subseteq V$, $\exists Q^{\oplus}.C \in \mathcal{L}(x)$ and $\mathbf{T} = (V, E, \mathcal{L}, x_{\mathbf{T}}, \dot{\neq})$ is a completion tree.

Input : A $\mathcal{SHIQ}_+$ knowledge base $(\mathcal{T}, \mathcal{R})$ and a $\mathcal{SHIQ}_+$-concept D

Output: Is D satisfiable w.r.t. $(\mathcal{T}, \mathcal{R})$?

1 Let $\mathbf{T} = (V, E, \mathcal{L}, x_{\mathbf{T}}, \dot{\neq})$ be an initial tree such that $V = \{x_{\mathbf{T}}\}$, $\mathcal{L}(x_{\mathbf{T}}) = \{D\}$, and there is no $x, y \in V$ such that $x \dot{\neq} y$;

2 **while** *there is a non-empty set* S *of expansion rules in Figure 2 and 3 such that each* r ∈ S *can be applied to a node* $x \in V$ **do**

3   Apply r ;

4 **if** *there is a clash-free tree* $\mathbf{T}'$ *which is built by Line 1 to 3* **then**

5   YES ;

6 **else**

7   NO ;

Algorithm 2: Algorithm for building a completion tree for a $\mathcal{SHIQ}_+$-concept w.r.t. a $\mathcal{SHIQ}_+$ knowledge base

1. Applications of rules in Figure 2 and 3 do not remove concepts from the label of nodes. Moreover, applications of rules in Figure 2 and 3 do not remove roles from the label of edges except that they may set the label of edges to an empty set. However, when the label of an edge becomes empty it remains to be empty forever. Therefore, we can compute a upper bound of the completion tree's height from the blocking condition. This upper bound equals $K = 2^{2m+k}$ where $m = \mathsf{card}\{\mathsf{sub}(\mathcal{T},\mathcal{R}) \cup \widehat{\mathsf{sub}}(\mathcal{T},\mathcal{R})\}$ and k is the number of roles occurring in $\mathcal{T}$ and $\mathcal{R}$ plus their inverse and transitive closure. Moreover, the number of neighbors of any node is bounded by $M = \sum m_i$ where m_i occurs in a number restriction term $(\geq m_i R.C)$ that appears in $\mathcal{T}$.
2. Algorithm 1 checks the clash-freeness of the third kind for each $x \in V$ with $\exists Q^{\oplus}.C \in \mathcal{L}(x)$. To do this, it starts from x and go up to an ancestor x' of x, and go down to a descendant of x' through the function $\mathsf{succ}(x')$. The length of such a path is bounded by $K \times L$ where K is given above and L is the number of blocked nodes of the completion tree. Algorithm 1 may consider all paths which go though all possible blocked nodes. The cardinality of this set is bounded by the number of all permutations of the blocked nodes. Therefore, the complexity of Algorithm 1 is bounded by $(K \times L) \times L!$. Algorithm 1 would be called for each occurrence of each term such as $\exists Q^{\oplus}.C$ that occurs in each node $v \in V$.

Lemma 3 (Soundness). *Let $(\mathcal{T},\mathcal{R})$ be a $\mathcal{SHIQ}_+$ knowledge base. Let D be a $\mathcal{SHIQ}_+$-concept w.r.t. $(\mathcal{T},\mathcal{R})$. If Algorithm 2 can build a clash-free completion tree for D w.r.t. $(\mathcal{T},\mathcal{R})$ then there is a tableau for D w.r.t. $(\mathcal{T},\mathcal{R})$.*

Proof. Assume that $\mathbf{T} = (V, E, \mathcal{L}, x_{\mathbf{T}}, \dot{\neq})$ is a clash-free completion tree for D w.r.t. $(\mathcal{T},\mathcal{R})$. First, we build an extended tree $\widehat{\mathbf{T}} = (\widehat{V}, \widehat{E}, \mathcal{L}, x_{\widehat{\mathbf{T}}}, \dot{\neq})$ from $\mathbf{T}$ with help of functions $\widehat{\mathsf{succ}}$ and $\mathsf{b}(x)$:

- $x_{\widehat{\mathbf{T}}} = x_{\mathbf{T}}$,
- If $x \in \widehat{V}$ and $x' \in \widehat{\mathsf{succ}}(x)$ then $x' \in \widehat{V}$. In particular, if z, z' are two disctint successors of x such that $\mathsf{b}(z) = \mathsf{b}(z')$ then $\mathsf{b}(z) \neq \mathsf{b}(z')$ in $\widehat{\mathsf{succ}}^1_R(x)$.

From the construction, it follows that if $s, s' \in \mathbf{S}$ and $s \neq s'$ then $\widehat{\mathsf{succ}}^1(s) \cap \widehat{\mathsf{succ}}^1(s') = \emptyset$. We define a tableau $T = (\mathbf{S}, \mathcal{L}', \mathcal{E})$ for D as follows:

- We define $\mathbf{S} = \bigcup_{n\geq 0} \widehat{\mathsf{succ}}^n(x_{\mathbf{T}})$. Note that $\mathbf{S}$ can be considered as the nodes of an extended tree of $\mathbf{T}$ defined by using $\widehat{\mathsf{succ}}$.
- For each $s \in \widehat{\mathsf{succ}}^n(x_{\mathbf{T}})$ there is a unique $x_s \in V$ such that $x_s \in \mathsf{succ}^k(x_{\mathbf{T}})$ and $s \in \widehat{\mathsf{succ}}^l(x_s)$ with $n = k + l$. We define $\mathcal{L}'(s) = \mathcal{L}(x_s)$.
- $\mathcal{E}(R) = \mathcal{E}_1(R) \cup \mathcal{E}_2(R)$ where
  $\mathcal{E}_1(R) = \{\langle s,t\rangle \in \mathbf{S}^2 \mid R \in \mathcal{L}(\langle x_s, x_t\rangle) \vee R^{\ominus} \in \mathcal{L}(\langle x_t, x_s\rangle)\}$, and
  $\mathcal{E}_2(R) = \{\langle s,t\rangle \in \mathbf{S}^2 \mid (R \in \mathcal{L}(\langle x_s, z\rangle) \wedge (\mathsf{b}(z) = x_t)) \vee (R^{\ominus} \in \mathcal{L}(\langle x_t, z'\rangle)) \wedge (\mathsf{b}(z') = x_s))\}$

We now show T satisfies all properties in Definition 3.

– P1-P5 hold due to the non-applicable of $\sqsubseteq$-rule, $\sqcap$-rule and $\sqcup$-rule in Figure 2 and the facts that $\mathbf{T}$ is clash-free.
– For P6, assume that $s, t \in \mathbf{S}$ with $\forall S.C \in \mathcal{L}'(s)$, $Q^{\oplus} \underline{\circledast} S$ and $\langle s, t\rangle \in \mathcal{E}(Q)$. By the definition of T, x_t is a $Q^{\oplus}$-neighbor of x_s. Due to the non-applicable of $\forall_+$-rule, it follows that $\forall Q^{\oplus}.C \in \mathcal{L}(x_t)$. By the definition of T, we have $\forall Q^{\oplus}.C \in \mathcal{L}'(t)$.
– For P7, assume that $s \in \mathbf{S}$ with $\exists S.C \in \mathcal{L}'(s)$. Due to the non-applicable of $\exists$-rule, it follows that x_s has a S-neighbor x_t such that $C \in \mathcal{L}(x_t)$. By the definition of T, we have $\langle s, t\rangle \in \mathcal{E}(S)$ and $C \in \mathcal{L}'(t)$.
– For P8, assume that $s \in \mathbf{S}$ with $\exists Q^{\oplus}.C \in \mathcal{L}'(s)$. Since $\mathbf{T}$ is clash-free (third kind), x_s has a $\exists Q^{\oplus}.C$-reachable x_n i.e. there are $x_1, \cdots, x_n$ such that x_{i+1} is a Q-neighbor of x_i or x_{i+1} blocks a Q-successor of x_i with $x_s = x_0$ and $\exists Q.C \in \mathcal{L}(x_n)$, $\exists Q^{\oplus}.C \in \mathcal{L}(x_i)$ for all $i \in \{0, \cdots, n-1\}$.
Assume that $\exists Q.C \in \mathcal{L}'(s)$. This implies that x_s has a Q-neighbor y such that $C \in \mathcal{L}(y)$ due to the non-applicable of $\exists$-rule. By the definition of T, there is some $t \in \mathbf{S}$ with $t \in \widehat{\mathsf{succ}}^1(s)$ or $s \in \widehat{\mathsf{succ}}^1(t)$ such that $\langle s, t\rangle \in \mathcal{E}(Q)$. Thus, P8 holds.
Assume that $\exists Q.C \notin \mathcal{L}'(s)$. According to the definition of $\exists Q^{\oplus}.C$-reachable nodes, there is some $0 \leq k < n$ such that x_k is an ancestor of x_0 and x_{k+1} is a (extended) successor of x_k. If $k = 0$ then there are $s_1, \cdots, s_n$ with $x_{s_i} = x_i$, $s_0 = s$ and $\langle s_i, s_{i+1}\rangle \in \mathcal{E}(Q)$, $\exists Q.C \in \mathcal{L}'(s_n)$, $\exists Q^{\oplus}.C \in \mathcal{L}'(s_i)$ for all $i \in \{0, \cdots, n\}$. Thus, P8 holds.
Assume that $k > 0$. We define a function $\widehat{\mathsf{pred}}^j(t)$ as follows: $\widehat{\mathsf{pred}}^j(t) = x_{\mathbf{T}}$ iff $t \in \widehat{\mathsf{succ}}^j(x_{\mathbf{T}})$ for all $t \in \mathbf{S}$. This implies that for each $t \in \mathbf{S}$ there is a unique j such that $\widehat{\mathsf{pred}}^j(t) = x_{\mathbf{T}}$. Let $x_{\mathbf{T}} = \widehat{\mathsf{pred}}^l(s)$, $x_{\mathbf{T}} = \widehat{\mathsf{pred}}^m(x_0) = \mathsf{pred}^m(x_0)$ and $x_{\mathbf{T}} = \widehat{\mathsf{pred}}^p(x_k) = \mathsf{pred}^p(x_k)$. We consider the following cases :
  1. Assume $m = l$. By the definition of T there are $s_0, \cdots, s_n \in \mathbf{S}$ such that $x_{s_i} = x_i$ and $\langle s_i, s_{i+1}\rangle \in \mathcal{E}(Q)$, $\exists Q^{\oplus}.C \in \mathcal{L}'(s_i)$ for all $i \in \{0, \cdots, n-1\}$ with $s_0 = s$ and $\exists Q.C \in \mathcal{L}'(s_n)$. Thus, P8 holds.
  2. Assume $m < l$. Let $0 \leq K \leq l$ be the least number such that $x_{\widehat{\mathsf{pred}}^K(s)}$ has a $\exists Q^{\oplus}.C$-reachable y with $y \in \widehat{\mathsf{succ}}^{K'}(x_{\widehat{\mathsf{pred}}^K(s)})$. We can pick $K = l - p$ with $x_{\mathbf{T}} = \widehat{\mathsf{pred}}^p(x_k)$ if there is no such K such that $K < l - p$. If $K = 0$ then $k = 0$, which is considered. For $K > 0$, we show that $\langle \widehat{\mathsf{pred}}^j(s), \widehat{\mathsf{pred}}^{j+1}(s)\rangle \in \mathcal{E}(Q)$ and $\exists Q^{\oplus}.C \in \mathcal{L}'(\widehat{\mathsf{pred}}^{j+1}(s))$ for all $j \in \{0, \cdots, K-1\}$ (***).
  For $j = 0$, we have $\langle s, \widehat{\mathsf{pred}}^1(s)\rangle \in \mathcal{E}(Q)$ and $\exists Q^{\oplus}.C \in \mathcal{L}'(\widehat{\mathsf{pred}}^1(s))$, since $\langle s, \widehat{\mathsf{pred}}^1(s)\rangle \notin \mathcal{E}(Q)$ or $\exists Q^{\oplus}.C \notin \mathcal{L}'(\widehat{\mathsf{pred}}^1(s))$ implies $K = 0$.
  Assume that $\exists Q^{\oplus}.C \in \mathcal{L}'(\widehat{\mathsf{pred}}^j(s))$ with $j < K$. Due to the clash-freeness (third kind) of $\mathbf{T}$, $x_{\widehat{\mathsf{pred}}^j(s)}$ has a $\exists Q^{\oplus}.C$-reachable node w i.e. there are nodes $w_1, \cdots, w_{n'}$ and some $k' \geq 0$ such that $w_{k'}$ is an ancestor of $x_{\widehat{\mathsf{pred}}^j(s)}$, $w_{k'+1}$ is a (extended) successor of w_k, and w_i is a Q-neighbor of w_{i+1} and $\exists Q^{\oplus}.C \in \mathcal{L}(w_i)$, $\exists Q.C \in \mathcal{L}(w_{n'})$ for all $i \in \{0, \cdots, n'-1\}$ with $w_0 = x_{\widehat{\mathsf{pred}}^j(s)}$. Due to $j < K$ and $\mathcal{L}'(\widehat{\mathsf{pred}}^{j+1}(s)) = \mathcal{L}(x_{\widehat{\mathsf{pred}}^{j+1}(s)})$, we have $k' > 0$ and $\exists Q^{\oplus}.C \in \mathcal{L}'(\widehat{\mathsf{pred}}^{j+1}(s))$. Thus, (***) holds.

From (***), it follows that there are $s_i = \widehat{\mathsf{pred}}^i(s)$ for all $i \in \{0, \cdots, K\}$ and $s_{K+j} = \widehat{\mathsf{succ}}^j(\widehat{\mathsf{pred}}^K(s))$ for all $j \in \{1, \cdots, K'\}$ such that $\langle s_h, s_{h+1}\rangle \in \mathcal{E}(Q)$ and $\exists Q^{\oplus}.C \in \mathcal{L}'(s_h)$, $\exists Q.C \in \mathcal{L}'(s_{K+K'})$ for all $h \in \{0, \cdots, K+K'\}$ with $s_0 = s$. Thus, P8 holds.

- For P9, assume that $s, t \in \mathbf{S}$ with $\langle s, t\rangle \in \mathcal{E}(Q^{\oplus})$. By the construction of T, x_t is a $Q^{\oplus}$-successor of x_s, or x_t blocks a $Q^{\oplus}$-successor y of x_s with $\mathcal{L}(y) = \mathcal{L}(x_t)$, or x_t is a $Q^{\oplus}$-predecessor of x_s, or x_s blocks a $(Q^{\ominus})^{\oplus}$-successor z of x_t with $\mathcal{L}(z) = \mathcal{L}(x_s)$, $\mathcal{L}(z') = \mathcal{L}(x_t)$ and $\mathcal{L}(\langle x_t, z\rangle) = \mathcal{L}(\langle z', x_s\rangle)$, $\langle z', x_s\rangle \in E$. Due to the non-applicable of $\oplus$-rule to x_s and x_t (or to x_s and y, or to z' and x_s), we have $\exists Q^{\oplus}.\Phi_\sigma \in \mathcal{L}(x_s)$ with $\sigma = \mathcal{L}(x_t) \cap \mathsf{sub}(\mathcal{T}, \mathcal{R})$. By the definition of T, it follows that $\exists Q^{\oplus}.\Phi_\sigma \in \mathcal{L}'(s)$ with $\sigma = \mathcal{L}'(t) \cap \mathsf{sub}(\mathcal{T}, \mathcal{R})$.
- P10 and P11 are consequences of the construction of $\mathbf{T}$.
- For P12 assume that $s \in \mathbf{S}$ with $(\geq nS.C) \in \mathcal{L}'(s)$. Due to the non-applicable of $\geq$-rule, x_s has n nodes $x_1, \cdots, x_n$ where x_i is S-neighbor of x_s or the blocking node of S-successor of x_s such that $C \in \mathcal{L}(x_i)$ for $i \in \{1, \cdots, n\}$. Due to the definition of $\mathbf{S}$, there are $s_1, \cdots, s_n \in \mathbf{S}$ with $x_i = x_{s_i}$, $\langle s, s_i\rangle \in \mathcal{E}(S)$ and $C \in \mathcal{L}'(s_i)$ for all $i \in \{1, \cdots, n\}$ such that $s_i \in \widehat{\mathsf{succ}}(s)$ for all $i \in \{1, \cdots, n\}$, or there is some $k \in \{1, \cdots, n\}$ such that $s \in \widehat{\mathsf{succ}}(s_k)$ and $s_i \in \widehat{\mathsf{succ}}(s)$ for all $i \in \{1, \cdots, n\}$ with $i \neq k$. By construction, we have $s_i \neq s_j$ with $i \neq j$.
- For P13 assume that $s \in \mathbf{S}$ with $(\leq nS.C) \in \mathcal{L}'(s)$. By absurdity, assume that there are $s_1, \cdots, s_{n+1} \in \mathbf{S}$ such that $\langle s, s_i\rangle \in \mathcal{E}(S)$ and $C \in \mathcal{L}'(s_i)$ for all $i \in \{1, \cdots, n+1\}$.
  If x_s has no blocked successor and $s_i \neq s_j$ then $x_{s_i} \neq x_{s_j}$. Assume that x_s has a blocked successor z. This implies that there is some s_k with $k \in \{1, \cdots, n\}$ such that $x_{s_k} = \mathsf{b}(z)$. If $s_i \neq s_j$ and z, z' are blocked with $x_{s_i} = \mathsf{b}(z)$, $x_{s_j} = \mathsf{b}(z')$ then, by the definition of $\widehat{\mathsf{succ}}$, $z \neq z'$. Due to the non-applicable of $\leq$-rule, x_s has $(n+1)$ distinct S-neighbors $x_1, \cdots, x_{(n+1)}$ such that $C \in \mathcal{L}(x_i)$ for $i \in \{1, \cdots, n\}$, which contradicts the clash-freeness (second kind).
- For P14 assume that $s, t \in \mathbf{S}$ with $(\leq nS.C) \in \mathcal{L}'(s)$ and $\langle s, t\rangle \in \mathcal{E}(S)$. By the definition of T, x_t is a S-neighbor of x_s and $(\leq nS.C) \in \mathcal{L}(x_s)$. Due to the non-applicable of ch-rule, we have $C \in \mathcal{L}(x_t)$ or $\dot{\neg} C \in \mathcal{L}(x_t)$. Due to the definition of T, this implies $C \in \mathcal{L}'(t)$ or $\dot{\neg} C \in \mathcal{L}'(t)$.

Lemma 4 (Completeness). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHIQ}_+$ knowledge base. Let D be a $\mathcal{SHIQ}_+$-concept w.r.t. $(\mathcal{T}, \mathcal{R})$. If there is a tableau for D w.r.t. $(\mathcal{T}, \mathcal{R})$ then Algorithm 2 can build a clash-free completion tree for D w.r.t. $(\mathcal{T}, \mathcal{R})$.*

Proof. Let $T = (\mathbf{S}, \mathcal{L}', \mathcal{E})$ be a tableau for $(\mathcal{T}, \mathcal{R})$. Let $\mathbf{T} = (V, E, \mathcal{L}, x_{\mathbf{T}}, \dot{\neq})$ be a completion tree. We show that there exists a sequence of expansion rule applications such that it generates a clash-free completion tree (**).

We define a function π from V to $\mathbf{S}$ progressively over the construction of $\mathbf{T}$ such that it satisfies the following conditions, denoted by (*):

1. $\mathcal{L}(x) \subseteq \mathcal{L}'(\pi(x))$ for $x \in V$,
2. if y is a S-neighbor of x in $\mathbf{T}$ then $\langle \pi(x), \pi(y)\rangle \in \mathcal{E}(S)$,
3. $x \dot{\neq} y$ implies $\pi(x) \neq \pi(y)$,

4. if $\exists Q^{\oplus}.C \in \mathcal{L}(x)$ and $\exists Q.C \in \mathcal{L}'(\pi(x))$ then $\exists Q.C \in \mathcal{L}(x)$ for $x \in V$,

To prove (**), we have to show that (i) we can apply expansion rules such that the conditions in (*) are preserved, and (ii) if the conditions (*) are satisfied when constructing a completion tree by expansion rules then the obtained completion tree is clash-free.

Since T is a tableau there is a node $s \in \mathbf{S}$ such that $D \in \mathcal{L}'(s)$. A node $x \in V$ is created with $\pi(x) = s$ and $\mathcal{L}(x) = \{D\}$. Applications of $\sqsubseteq$-rule, $\sqcap$-rule, $\exists$-rule, $\forall$-rule, $\forall_+$-rule, $\leq$-rule, $\geq$-rule and ch-rule preserve the conditions in (*). The proof is similar to that in [11]. We now concentrate on $\exists_+$-rule, $\oplus$-rule and $\sqcup$-rule.

1. For $\exists_+$-rule, assume that $\exists Q^{\oplus}.C \in \mathcal{L}(x)$. Due to the condition 1 in (*) (induction hypothesis), we have $\exists Q^{\oplus}.C \in \mathcal{L}'(\pi(x))$. Due to P8 and P4, we have $(\exists Q.C \sqcup \exists Q.\exists Q^{\oplus}.C) \in \mathcal{L}'(\pi(x))$ and $\{\exists Q.C, \exists Q.\exists Q^{\oplus}.C\} \cap \mathcal{L}'(\pi(x)) \neq \emptyset$. Application of $\exists_+$-rule and $\sqcup$-rule to x yields $\{\exists Q.C, \exists Q.\exists Q^{\oplus}.C\} \cap \mathcal{L}(x) \neq \emptyset$. If $\exists Q.C \in \mathcal{L}'(\pi(x))$ then $\sqcup$-rule can be applied to x such that $\exists Q.C \in \mathcal{L}(x)$. Thus, the conditions in (*) are preserved.
2. For $\oplus$-rule, assume that $Q^{\oplus} \in \mathcal{L}(\langle x, y\rangle)$. Due to the condition 2 in (*) (induction hypothesis), we have $Q^{\oplus} \in \mathcal{L}'(\langle \pi(x), \pi(y)\rangle)$. Moreover, due to P9 we have $\exists P^{\oplus}.\Phi_\sigma \in \mathcal{L}'(\pi(x))$ with $\sigma = \mathcal{L}'(\pi(y)) \cap \mathsf{sub}(\mathcal{T}, \mathcal{R})$. $\oplus$-rule can be applied to $\langle x, y\rangle$ such that $\mathcal{L}(x) = \mathcal{L}(x) \cup \{\exists P^{\oplus}.\Phi_\sigma\}$ with $\sigma = \mathcal{L}'(\pi(y)) \cap \mathsf{sub}(\mathcal{T}, \mathcal{R})$. This implies that $\mathcal{L}(y) \subseteq \mathcal{L}'(\pi(y))$. Therefore, the conditions in (*) are preserved.

We show that if a completion tree $\mathbf{T}$ can be built with a function π satisfying (*) then $\mathbf{T}$ is clash-free.

1. If the condition 1 in (*) is satisfied then there is no node x in $\mathbf{T}$ such that $A, \dot{\neg} A \in \mathcal{L}(x)$ due to P2 and the condition 1. That means that T does not contain a clash of the first kind as described in Definition 4.
2. There is no clash of the second kind in $\mathbf{T}$ if the conditions 1 to 3 in (*) are satisfied with P12.
3. Assume that $\exists Q^{\oplus}.C \in \mathcal{L}(x)$. Due to the condition 1 in (*), we have $\exists Q^{\oplus}.C \in \mathcal{L}'(\pi(x))$. According to P8 and P4, there are $s_1, \cdots, s_n \in \mathbf{S}$ such that $\langle s_i, s_{i+1}\rangle \in \mathcal{E}(Q)$, $\exists Q^{\oplus}.C \in \mathcal{L}'(s_i)$ and $\{\exists Q.\exists Q^{\oplus}.C, \exists Q.C\} \cap \mathcal{L}'(s_i) \neq \emptyset$ for $i \in \{0, \cdots, n-1\}$ with $s_0 = \pi(x)$, and $\exists Q.C \in \mathcal{L}'(s) \cup \mathcal{L}'(s_{n-1})$.
Assume $\exists Q.C \in \mathcal{L}'(s)$. Due to the condition 4 in (*), we have $\exists Q.C \in \mathcal{L}(x)$. This implies that $\mathbf{T}$ does not have a clash of the third kind.
Assume $\exists Q.C \notin \mathcal{L}'(s)$ and $n > 1$. Without loss of the generality, assume that $\exists Q.C \notin \mathcal{L}'(s_i)$ for all $i \in \{0, \cdots, n-2\}$ and $\exists Q.C \in \mathcal{L}'(s_{n-1})$ (otherwise, if there is some $0 \leq k < n-1$ such that $\exists Q.C \in \mathcal{L}'(s_k)$ then we pick $n = k+1$).
By applying successively $\exists$-rule, $\exists_+$-rule and $\sqcup$-rule, there are nodes $x_1, \cdots, x_l \in V$ such that $\pi(x_i) = s_i$, $Q \in \mathcal{L}(\langle x_{i-1}, x_i\rangle)$ and $\{\exists Q^{\oplus}.C, \exists Q.\exists Q^{\oplus}.C\} \subseteq \mathcal{L}(x_i)$ for all $i \leq l$ with some $l \leq n-1$. If $l = n-1$ then x has a $\exists Q^{\oplus}.C$-reachable node x_l such that $\exists Q.C \in \mathcal{L}(x_l)$ due to $\exists Q^{\oplus}.C \in \mathcal{L}(x_l)$, $\exists Q.C \in \mathcal{L}'(\pi(x_l))$ and the condition 4 in (*). If $l < n-1$ and x_l is blocked by z then we restart from the node z with $\exists Q^{\oplus}.C \in \mathcal{L}(z)$ (since $\mathcal{L}(z) = \mathcal{L}(x_l)$) finding $x'_1, \cdots, x'_{l'} \in V$ which

have the same properties as those of $x_1, \cdots, x_l$. This process can be repeated until finding a node $w \in V$ such that w is a $\exists Q^{\oplus}.C$-reachable node of x.
Therefore, $\mathbf{T}$ does not have a clash of the third kind.

The following theorem is a consequence of Lemmas 2, 3 and 4.

Theorem 1. *Algorithm 2 is a decision procedure for satisfiability of $\mathcal{SHIQ}_+$-concepts w.r.t. $\mathcal{SHIQ}_+$ knowledge bases.*

4 $\mathcal{SHIN}_+$ is undecidable

This section show that if we add transitive closure of roles to the logic $\mathcal{SHIN}$ without restriction on role hierachies then the obtained logic, denoted by $\mathcal{SHIN}_+$, is undecidable. More precisely, $\mathcal{SHIN}_+$ role names must be simple according to the definition of simple roles for $\mathcal{SHIQ}$ i.e. each role name R has no transitive sub-role w.r.t. the role hierarchy. In addition, role axioms like $R \sqsubseteq S^+$ or $R^+ \sqsubseteq S$ are allowed in $\mathcal{SHIN}_+$ role hierarchies where R, S are role names and R^+ and S^+ are transitive closures of R and S respectively.

The undecidability proof uses a reduction of the domino problem [14]. The following definition, which is taken from [11], reformulates the problem in a more precise way.

Definition 5. *A domino system $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ consists of a non-empty set of domino types $\mathcal{D} = \{D_1, \cdots, D_l\}$ and of sets of horizontally and vertically matching pairs $\mathcal{H} \subseteq \mathcal{D} \times \mathcal{D}$ and $\mathcal{V} \subseteq \mathcal{D} \times \mathcal{D}$. The problem is to determine if, for a given $\mathbf{D}$, there exists a tiling of an $\mathbb{N} \times \mathbb{N}$ grid such that each point of the grid is covered with a domino type in $\mathcal{D}$ and all horizontally and vertically adjacent pairs of domino types are in $\mathcal{H}$ and $\mathcal{V}$ respectively, i.e., a mapping $t : \mathbb{N} \times \mathbb{N} \to \mathcal{D}$ such that for all $m, n \in \mathbb{N}$, $\langle t(m,n), t(m+1,n)\rangle \in \mathcal{H}$ and $\langle t(m,n), t(m,n+1)\rangle \in \mathcal{V}$.*

The reduction of the domino problem to the satisfiability of $\mathcal{SHIN}_+$-concepts will be carried out by (i) constructing a concept, namely A, and two sets of concept and role inclusion axioms, namely $\mathcal{T}_D$ and $\mathcal{R}_D$, and (ii) showing that the domino problem is equivalent to the satisfiability of A w.r.t. $\mathcal{T}_D$ and $\mathcal{R}_D$. Axioms in Definition 6 specify a grid (Fig.4) that represents such a domino system.

Globally, given a domino set $\mathcal{D} = \{D_1, \cdots, D_l\}$, we need axioms that impose that each point of the plane is covered by exactly one $D_i^{\mathcal{I}}$ (axiom 8 in Definition. 6) and ensure that each D_i is compatibly placed in the horizontal and vertical lines (axiom 9). Locally, the key idea is to use $\mathcal{SHIN}_+$ axioms for describing the grid as illustrated in Figure 5. For example, we consider how a square of the grid can be formed. Axiom 10 in Definition 6 says that if A has an instance $x_A^{\mathcal{I}}$ with an interpretation $\mathcal{I}$, then there are three instances $x_B^{\mathcal{I}}, x_C^{\mathcal{I}}, x_D^{\mathcal{I}}$ in $B^{\mathcal{I}}, C^{\mathcal{I}}, D^{\mathcal{I}}$, respectively, such that $\langle x_A^{\mathcal{I}}, x_B^{\mathcal{I}}\rangle \in {X_1^1}^{\mathcal{I}}$, $\langle x_A^{\mathcal{I}}, x_C^{\mathcal{I}}\rangle \in {Y_1^1}^{\mathcal{I}}$ and $\langle x_A^{\mathcal{I}}, x_D^{\mathcal{I}}\rangle \in {\varepsilon_{AD}}^{\mathcal{I}}$. These instances are distinct since A, B, C, D are disjoint by axioms 10, 11, 12 and 13. In addition, by axioms 11, 12, there are $x_D'^{\mathcal{I}}, x_D''^{\mathcal{I}} \in D^{\mathcal{I}}$ such that $\langle x_B^{\mathcal{I}}, x_D'^{\mathcal{I}}\rangle \in {Y_2^1}^{\mathcal{I}}$, $\langle x_C^{\mathcal{I}}, x_D''^{\mathcal{I}}\rangle \in {X_2^1}^{\mathcal{I}}$. This is depicted in Figure 5.

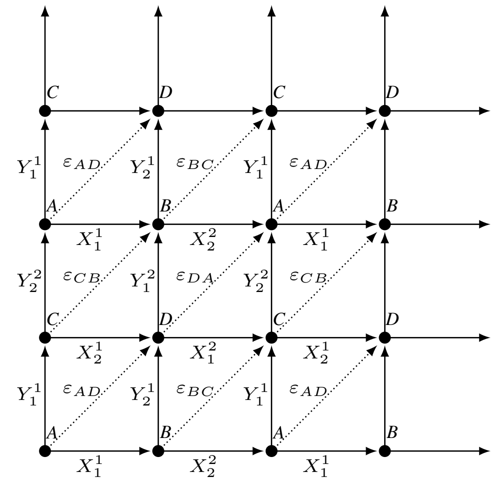

Fig. 4. The grid illustrates a model of the concept A w.r.t the axioms

Since P_{12}^{11} subsumes X_1^1, Y_2^1 by axiom 1, we have $\langle x_A^{\mathcal{I}}, x_D'^{\mathcal{I}} \rangle \in ({P_{12}^{11}}^+)^{\mathcal{I}}$. Moreover, since P_{12}^{11} is functional by axiom 5, $\langle x_A^{\mathcal{I}}, x_D^{\mathcal{I}} \rangle \in ({P_{12}^{11}}^+)^{\mathcal{I}}$ by axiom 3, and ${\varepsilon_{AD}}^{\mathcal{I}} \subseteq ({P_{12}^{11}}^+)^{\mathcal{I}}$ by axiom 3, there are two possibilities: (i) $x_D^{\mathcal{I}} = x_D'^{\mathcal{I}}$, and (ii) there is $y^{\mathcal{I}}$ such that $\langle x_D'^{\mathcal{I}}, y^{\mathcal{I}} \rangle \in (P_{12}^{11})^{\mathcal{I}}$. This contradicts axiom 13. Therefore, $x_D^{\mathcal{I}} = x_D'^{\mathcal{I}}$. Similarly, we can get $x_D^{\mathcal{I}} = x_D''^{\mathcal{I}}$. By this way, the axioms in Definition 6 can yield a grid as in Figure 5 if concept A is satisfiable.

Definition 6. *Let* $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ *be a domino system with* $\mathcal{D} = \{D_1, \cdots, D_l\}$. *Let* N_C *and* N_R *be sets of concept and role names such that* $N_C = \{A, B, C, D\} \cup \mathcal{D}$, $N_R = \{X_j^i \mid i, j \in \{1,2\}\} \cup \{X, Y\} \cup \{P_{rs}^{ij} \mid i, j, r, s \in \{1,2\}, r \neq s\} \cup \{\varepsilon_{AD}, \varepsilon_{DA}, \varepsilon_{BC}, \varepsilon_{CB}\}$

Role hierarchy:

1. $X_r^i \sqsubseteq P_{rs}^{ij}, Y_s^j \sqsubseteq P_{rs}^{ij}$ *for all* $i, j, r, s \in \{1,2\}, r \neq s$,
2. $X_r^i \sqsubseteq X, Y_r^i \sqsubseteq Y$ *for all* $i, r \in \{1,2\}$,
3. $\varepsilon_{AD} \sqsubseteq {P_{12}^{11}}^+, \varepsilon_{AD} \sqsubseteq {P_{21}^{11}}^+, \varepsilon_{DA} \sqsubseteq {P_{12}^{22}}^+, \varepsilon_{DA} \sqsubseteq {P_{21}^{22}}^+$,
4. $\varepsilon_{BC} \sqsubseteq {P_{21}^{21}}^+, \varepsilon_{BC} \sqsubseteq {P_{12}^{21}}^+, \varepsilon_{CB} \sqsubseteq {P_{21}^{12}}^+, \varepsilon_{CB} \sqsubseteq {P_{12}^{12}}^+$,

   Concept inclusion axioms:

5. $\top \sqsubseteq \leq 1 P_{rs}^{ij}$ *for all* $i, j, r, s \in \{1,2\}, r \neq s$,
6. $\top \sqsubseteq \ \leq 1 X, \top \sqsubseteq \ \leq 1 Y$,
7. $\top \sqsubseteq \leq 1 \varepsilon_{AD}$, $\top \sqsubseteq \leq 1 \varepsilon_{DA}$, $\top \sqsubseteq \leq 1 \varepsilon_{BC}$, $\top \sqsubseteq \leq 1 \varepsilon_{CB}$,
8. $\top \sqsubseteq \bigsqcup_{1 \leq i \leq l} (D_i \sqcap (\bigsqcap_{1 \leq j \leq l, j \neq i} \neg D_j))$,
9. $D_i \sqsubseteq \forall X. \bigsqcup_{(D_i, D_j) \in \mathcal{H}} D_j \sqcap \forall Y. \bigsqcup_{(D_i, D_k) \in \mathcal{V}} D_k$ *for each* $D_i \in \mathcal{D}$,

10. $A \sqsubseteq \neg B \sqcap \neg C \sqcap \neg D \sqcap \exists X_1^1.B \sqcap \exists Y_1^1.C \sqcap \exists \varepsilon_{AD}.D \sqcap \forall P_{12}^{22}.\bot \sqcap \forall P_{21}^{22}.\bot$,
11. $B \sqsubseteq \neg A \sqcap \neg C \sqcap \neg D \sqcap \exists X_2^2.A \sqcap \exists Y_2^1.D \sqcap \exists \varepsilon_{BC}.C \sqcap \forall P_{21}^{12}.\bot \sqcap \forall P_{12}^{12}.\bot$,
12. $C \sqsubseteq \neg A \sqcap \neg B \sqcap \neg D \sqcap \exists X_2^1.D \sqcap \exists Y_2^2.A \sqcap \exists \varepsilon_{CB}.B \sqcap \forall P_{21}^{21}.\bot \sqcap \forall P_{12}^{21}.\bot$,
13. $D \sqsubseteq \neg A \sqcap \neg B \sqcap \neg C \sqcap \exists X_1^2.C \sqcap \exists Y_1^2.B \sqcap \exists \varepsilon_{DA}.A \sqcap \forall P_{12}^{11}.\bot \sqcap \forall P_{21}^{11}.\bot$.

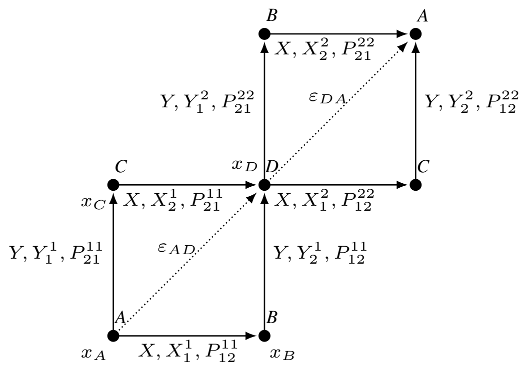

Fig. 5. How each square can be formed from a diagonal represented by an ε

Theorem 2 (Undecidability of $\mathcal{SHIN}_+$). *The concept A is satisfiable w.r.t. concept and role inclusion axioms in Definition 6 iff there is a compatible tiling t of the first quadrant $\mathbb{N} \times \mathbb{N}$ for a given domino system $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$.*

A proof of Theorem 2 can be found in Appendix.

5 Conclusion and Discussion

In this paper, we have presented a tableaux-based decision procedure for $\mathcal{SHIQ}_+$ concept satisfiability. In order to define tableaux for $\mathcal{SHIQ}_+$ we introduces new properties that allow us to represent semantic constraints imposed by transitive closure of roles and to avoid expressing explicitly cycles for role inclusion axioms with transitive closure. These new tableaux properties are translated into new non-deterministic expansion rules which make the complexity of the tableaux-based algorithm jump from exponential for $\mathcal{SHIQ}$ to double exponential for $\mathcal{SHIQ}_+$. An open issue consists in investigating whether this complexity is worst-case optimal. To the best of our knowledge, this problem has not addressed yet. Another future work concerns the extension of our tableaux-based algorithm to $\mathcal{SHIQ}_+$ with nominals.

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Appendix

Theorem (2) **[Undecidability of $\mathcal{SHIN}_+$]** *The concept A is satisfiable iff there is a compatible tiling t of the first quadrant $\mathbb{N} \times \mathbb{N}$ for a given domino system $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$.*

Proof of Theorem 2
• *"If-direction"*. Assume that there is a compatible tiling t for $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$. This tiling is used to define an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of the concept A w.r.t. the axioms in Definition 6. Without loss of the generality, we assume that $t(0,0) = A$. Moreover, each $a_{(m,n)}$ is denoted for each point (m,n) of the first quadrant $\mathbb{N} \times \mathbb{N}$. The figure 1. illustrates the interpretation that we expect.

1. $\Delta^{\mathcal{I}} = \{a_{(m,n)} \mid m, n \in \mathbb{N}\}$

2. $(X_1^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
3. $(X_2^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
4. $(X_2^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
5. $(X_1^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
6. $(Y_1^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
7. $(Y_2^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
8. $(Y_2^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
9. $(Y_1^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
10. $(P_{12}^{11})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
11. $(P_{21}^{11})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
12. $(P_{12}^{22})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
13. $(P_{21}^{22})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
14. $(P_{21}^{21})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
15. $(P_{12}^{21})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
16. $(P_{21}^{12})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
17. $(P_{12}^{12})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\} \cup \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
18. $(\varepsilon_{AD})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
19. $(\varepsilon_{DA})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
20. $(\varepsilon_{BC})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
21. $(\varepsilon_{BC})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
22. ${D_i}^{\mathcal{I}} = \{a_{(k,l)} \mid t(k,l) = D_i\}$ for each $D_i \in \mathcal{D}$
23. $A^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
24. $D^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
25. $B^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
26. $C^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
27. $X^{\mathcal{I}} = {X_1^1}^{\mathcal{I}} \cup {X_2^1}^{\mathcal{I}} \cup {X_1^2}^{\mathcal{I}} \cup {X_2^2}^{\mathcal{I}}$
28. $Y^{\mathcal{I}} = {Y_1^1}^{\mathcal{I}} \cup {Y_2^1}^{\mathcal{I}} \cup {Y_1^2}^{\mathcal{I}} \cup {Y_2^2}^{\mathcal{I}}$

We now check that $\mathcal{I}$ satisfies all axioms in Definition 6.

1. $X_r^i \sqsubseteq P_{rs}^{ij}, Y_s^j \sqsubseteq P_{rs}^{ij}$ for all $i, j, r, s \in \{1, 2\}, r \neq s$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 2, 6, we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_1^1}^{\mathcal{I}}$, and $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_1^1}^{\mathcal{I}}$. From the assertions 10, 11 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {P_{12}^{11}}^{\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {P_{21}^{11}}^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.

2. $X_r^i \sqsubseteq X, Y_r^i \sqsubseteq Y$ for all $i, r \in \{1, 2\}$. From assertions 27 and 28.
3. $\varepsilon_{AD} \sqsubseteq (P_{12}^{11})^+, \varepsilon_{AD} \sqsubseteq (P_{21}^{11})^+$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertion 18, we have $\langle a_{(k,l)}, a_{(k+1,l+1)} \rangle \in \varepsilon_{AD}{}^{\mathcal{I}}$. From the assertions 2 and 8 it follows that $\langle a_{(k,l)}, a_{(k+1,l)} \rangle \in {X_1^1}^{\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)} \rangle \in {Y_2^1}^{\mathcal{I}}$ (note that $(k+1 \bmod 2 = 1)$). By the assertion 10 we have $\langle a_{(k,l)}, a_{(k+1,l)} \rangle \in {P_{12}^{11}}^{\mathcal{I}}$ and $\langle a_{(k+1,l)}, a_{(k+1,l+1)} \rangle \in {P_{12}^{11}}^{\mathcal{I}}$. This implies that $\langle a_{(k,l)}, a_{(k+1,l+1)} \rangle \in (P_{12}^{11})^{+\mathcal{I}}$.
     On the other hand, from the assertions 6 and 4 we have $\langle a_{(k,l)}, a_{(k,l+1)} \rangle \in {Y_1^1}^{\mathcal{I}}$, $\langle a_{(k,l+1)}, a_{(k+1,l+1)} \rangle \in {X_2^1}^{\mathcal{I}}$ (note that $(l + 1 \bmod 2 = 1)$). By the assertion 11 we have $\langle a_{(k,l)}, a_{(k,l+1)} \rangle \in {P_{21}^{11}}^{\mathcal{I}}$, $\langle a_{(k,l+1)}, a_{(k+1,l+1)} \rangle \in {P_{21}^{11}}^{\mathcal{I}}$. This implies that $\langle a_{(k,l)}, a_{(k+1,l+1)} \rangle \in (P_{21}^{11})^{+\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
4. $\varepsilon_{DA} \sqsubseteq (P_{12}^{22})^+, \varepsilon_{DA} \sqsubseteq (P_{21}^{22})^+$. Similarly.
5. $\varepsilon_{BC} \sqsubseteq (P_{21}^{21})^+, \varepsilon_{BC} \sqsubseteq (P_{12}^{21})^+$. Similarly.
6. $\varepsilon_{CB} \sqsubseteq (P_{21}^{12})^+, \varepsilon_{CB} \sqsubseteq (P_{12}^{12})^+$. Similarly.
7. $\top \sqsubseteq \ \leq 1 P_{r,s}^{ij}$ for all $i, j, r, s \in \{1, 2\}, r \neq s$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 10, 11, 14, 17 we have $\langle a_{(k,l)}, a_{(k+1,l)} \rangle \in {P_{12}^{11}}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)} \rangle \in {P_{21}^{11}}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)} \rangle \in {P_{21}^{21}}^{\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k+1,l)} \rangle \in {P_{12}^{12}}^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
8. $\top \sqsubseteq \ \leq 1X, \top \sqsubseteq \ \leq 1Y$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 2, 6 we have $\langle a_{(k,l)}, a_{(k+1,l)} \rangle \in {X_1^1}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)} \rangle \in {Y_1^1}^{\mathcal{I}}$. From the assertion 28, we have $\langle a_{(k,l)}, a_{(k+1,l)} \rangle \in X^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)} \rangle \in Y^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
9. $\top \sqsubseteq \ \leq 1\varepsilon_{AD}$. It is obvious from the assertion 18 for each $k, l \geq 0$.
10. $\top \sqsubseteq \ \leq 1\varepsilon_{DA}$. It is obvious from the assertion 19 for each $k, l \geq 0$.
11. $\top \sqsubseteq \ \leq 1\varepsilon_{BC}$. It is obvious from the assertion 20 for each $k, l \geq 0$.
12. $\top \sqsubseteq \ \leq 1\varepsilon_{CB}$. It is obvious from the assertion 21 for each $k, l \geq 0$.
13. $\top \sqsubseteq \bigsqcup_{1 \leq i \leq l} (D_i \sqcap (\bigsqcap_{1 \leq j \leq l, j \neq i} \neg D_j))$. Since t is a tiling, each (k, l) has a unique $D_i \in \mathcal{D}$ such that $t(k, l) = D_i$. Thus, from the assertion 22, each $a_{(k,l)}$ has a unique $D_i \in \mathcal{D}$ such that $a_{(k,l)} \in {D_i}^{\mathcal{I}}$.

14. $D_i \sqsubseteq \forall X. \bigsqcup_{(D_i,D_j)\in\mathcal{H}} D_j \sqcap \forall Y. \bigsqcup_{(D_i,D_k)\in\mathcal{V}} D_k$ for each $D_i \in \mathcal{D}$.
From the assertion 22, if $a_{(k,l)} \in D_i{}^{\mathcal{I}}$ then $t(k,l) = D_i$. Since t is a tiling, according to Definition 5 we have $\langle D_i, D_j\rangle \in \mathcal{H}$ and $\langle D_i, D_k\rangle \in \mathcal{V}$ with $t(k+1,l) = D_j$ and $t(k,l+1) = D_k$. From the assertions 28 and 2-9 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X^{\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y^{\mathcal{I}}$. From the assertion 22, we have $a_{(k+1,l)} \in D_j{}^{\mathcal{I}}$ and $a_{(k,l+1)} \in D_k{}^{\mathcal{I}}$.
15. $A \sqsubseteq \neg B \sqcap \neg C \sqcap \neg D \sqcap \exists X_1^1.B \sqcap \exists Y_1^1.C \sqcap \exists \varepsilon_{AD}.D \sqcap \forall P_{12}^{22}.\bot \sqcap \forall P_{21}^{22}.\bot$.
For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 23 we have $a_{(k,l)} \in A^{\mathcal{I}}$. From the assertions 24, 25, 26, we have $a_{(k,l)} \notin B^{\mathcal{I}}, a_{(k,l)} \notin C^{\mathcal{I}}, a_{(k,l)} \notin D^{\mathcal{I}}$. Moreover, from the assertions 2, 6 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X_1^{1\,\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y_1^{1\,\mathcal{I}}$. By the assertions 18 and 24 we have $\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \in \varepsilon_{AD}{}^{\mathcal{I}}$ and $a_{(k+1,l+1)} \in D^{\mathcal{I}}$.
     Additionally, according to the assertions 12, 13, $\langle a_{(k,l)}, a_{(k+1,l)}\rangle, \langle a_{(k,l)}, a_{(k,l+1)}\rangle \notin P_{12}^{11\,\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k+1,l)}\rangle, \langle a_{(k,l)}, a_{(k,l+1)}\rangle \notin P_{21}^{11\,\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. From the assertion 23, it follows $a_{(k,l)} \notin A^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
16. $B \sqsubseteq \neg A \sqcap \neg C \sqcap \neg D \sqcap \exists X_2^2.A \sqcap \exists Y_2^1.D \sqcap \exists \varepsilon_{BC}.C \sqcap \forall P_{21}^{12}.\bot \sqcap \forall P_{12}^{12}.\bot$. Similarly.
17. $C \sqsubseteq \neg A \sqcap \neg B \sqcap \neg D \sqcap \exists X_2^1.D \sqcap \exists Y_2^2.A \sqcap \exists \varepsilon_{CB}.B \sqcap \forall P_{21}^{21}.\bot \sqcap \forall P_{12}^{21}.\bot$. Similarly.
18. $D \sqsubseteq \neg A \sqcap \neg B \sqcap \neg C \sqcap \exists X_1^2.C \sqcap \exists Y_1^2.B \sqcap \exists \varepsilon_{DA}.A \sqcap \forall P_{12}^{11}.\bot \sqcap \forall P_{21}^{11}.\bot$. Similarly.

• *"Only-If-direction"*. On the other hand, assume that the concept A is satisfiable w.r.t. the axioms in Definition 6 , and let $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}}\rangle$ be an interpretation such that $A^{\mathcal{I}} \neq \emptyset$. Assume that $a_{(0,0)} \in A^{\mathcal{I}}$. This interpretation can be used to find a compatible tiling for **D**.

First, we show the following claim:

Claim. There are individuals $a_{(k,l)} \in \Delta^{\mathcal{I}}$ with $k, l \geq 0$ such that

- If $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$ then $a_{(k,l)} \in A^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in B^{\mathcal{I}}, a_{(k,l+1)} \in C^{\mathcal{I}}, a_{(k+1,l+1)} \in D^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X_1^{1\,\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in Y_2^{1\,\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y_1^{1\,\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in X_2^{1\,\mathcal{I}}$.
- If $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$ then $a_{(k,l)} \in D^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in C^{\mathcal{I}}, a_{(k,l+1)} \in B^{\mathcal{I}}, a_{(k+1,l+1)} \in A^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X_1^{2\,\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in Y_2^{2\,\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y_1^{2\,\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in X_2^{2\,\mathcal{I}}$.
- If $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$ then $a_{(k,l)} \in B^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in A^{\mathcal{I}}, a_{(k,l+1)} \in D^{\mathcal{I}}, a_{(k+1,l+1)} \in C^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X_2^{2\,\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in Y_1^{1\,\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y_2^{1\,\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in X_1^{2\,\mathcal{I}}$.

- If $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$ then $a_{(k,l)} \in C^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in D^{\mathcal{I}}, a_{(k,l+1)} \in A^{\mathcal{I}}, a_{(k+1,l+1)} \in B^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X_2^{1\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in Y_1^{2\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y_2^{2\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in X_1^{1\mathcal{I}}$.

Proof (Proof of the claim 5).

- Assume $k = 0, l = 0$. We have $a_{(0,0)} \in A^{\mathcal{I}}$. By the axiom 10 in Definition 6 there are $a_{(1,0)} \in B^{\mathcal{I}}, a_{(0,1)} \in C^{\mathcal{I}}$ such that $\langle a_{(0,0)}, a_{(1,0)}\rangle \in X_1^{1\mathcal{I}}$, $\langle a_{(0,0)}, a_{(0,1)}\rangle \in Y_1^{1\mathcal{I}}$. Moreover, by the axioms 11, 12 in Definition 6 there are $a_{(1,1)}, a'_{(1,1)} \in D^{\mathcal{I}}$ such that $\langle a_{(1,0)}, a_{(1,1)}\rangle \in Y_2^{1\mathcal{I}}$, $\langle a_{(0,1)}, a'_{(1,1)}\rangle \in X_2^{1\mathcal{I}}$. We show that $a'_{(1,1)} = a_{(1,1)}$.
  By the axiom 10 in Definition 6, let $a \in D^{\mathcal{I}}$ such that $\langle a_{(0,0)}, a\rangle \in \varepsilon_{AD}^{\mathcal{I}}$. From the axiom 1 in Definition 6 we have $\langle a_{(0,0)}, a_{(1,0)}\rangle, \langle a_{(1,0)}, a_{(1,1)}\rangle \in P_{12}^{11\mathcal{I}}$. If $a_{(1,1)} \neq a$ then, by the axioms 3, 5 in Definition 6 there is an instance a' such that $\langle a_{(1,1)}, a'\rangle \in P_{12}^{11\mathcal{I}}$, which contradicts the axiom 13 in Definition 6 since $a_{(1,1)} \in D^{\mathcal{I}}$ and $\langle a_{(1,1)}, a'\rangle \in P_{12}^{11\mathcal{I}}$. Thus, $a_{(1,1)} = a$. Analogously, from the axiom 1 in Definition 6 we have $\langle a_{(0,0)}, a_{(0,1)}\rangle, \langle a_{(0,1)}, a'_{(1,1)}\rangle \in P_{21}^{11\mathcal{I}}$. If $a'_{(1,1)} \neq a$ then, by the axioms 3, 5 in Definition 6 there is an instance a'' such that $\langle a'_{(1,1)}, a''\rangle \in P_{21}^{11\mathcal{I}}$, which contradicts the axiom 13 in Definition 6 since $a'_{(1,1)} \in D^{\mathcal{I}}$ and $\langle a'_{(1,1)}, a''\rangle \in P_{21}^{11\mathcal{I}}$. Therefore, $a'_{(1,1)} = a$, and thus $a_{(1,1)} = a'_{(1,1)}$.
- Assume that $k \geq 0$ or $l \geq 0$. We consider the following cases:
  - Assume $a_{(k,l)} \in A^{\mathcal{I}}$ with $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. By the axiom 10 in Definition 6 there are $a_{(k+1,l)} \in B^{\mathcal{I}}, a_{(k,l+1)} \in C^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X_1^{1\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y_1^{1\mathcal{I}}$. Moreover, by the axioms 11, 12 in Definition 6 there are $a_{(k+1,l+1)}, a'_{(k+1,l+1)} \in D^{\mathcal{I}}$ such that $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in Y_2^{1\mathcal{I}}$, $\langle a_{(k,l+1)}, a'_{(k+1,l+1)}\rangle \in X_2^{1\mathcal{I}}$. We show that $a'_{(k+1,l+1)} = a_{(k+1,l+1)}$.
    By the axiom 10 in Definition 6, let $a \in D^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a\rangle \in \varepsilon_{AD}^{\mathcal{I}}$. From the axiom 1 in Definition 6 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle, \langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in P_{12}^{11\mathcal{I}}$. If $a_{(k+1,l+1)} \neq a$ then, by the axioms 3, 5 in Definition 6 there is an instance a' such that $\langle a_{(k+1,l+1)}, a'\rangle \in P_{12}^{11\mathcal{I}}$, which contradicts the axiom 13 in Definition 6 since $a_{(k+1,l+1)} \in D^{\mathcal{I}}$ and $\langle a_{(k+1,l+1)}, a'\rangle \in P_{12}^{11\mathcal{I}}$. Thus, $a_{(k+1,l+1)} = a$. Analogously, from the axiom 1 in Definition 6 we have $\langle a_{(k,l)}, a_{(k,l+1)}\rangle, \langle a_{(k,l+1)}, a'_{(k+1,l+1)}\rangle \in P_{21}^{11\mathcal{I}}$. If $a'_{(k+1,l+1)} \neq a$ then, by the axioms 3, 5 in Definition 6 there is an instance a'' such that $\langle a'_{(k+1,l+1)}, a''\rangle \in P_{21}^{11\mathcal{I}}$, which contradicts the axiom 13 in Definition 6 since $a'_{(k+1,l+1)} \in D^{\mathcal{I}}$ and $\langle a'_{(k+1,l+1)}, a''\rangle \in P_{21}^{11\mathcal{I}}$. Therefore, $a'_{(k+1,l+1)} = a$, and thus $a_{(k+1,l+1)} = a'_{(k+1,l+1)}$.
    Obviously, if $(k \bmod 2 = 0)$ and $(l \bmod 2 = 0)$ then $((k+1) \bmod 2 = 1)$ and $((l+1) \bmod 2 = 1)$

- Assume $a_{(k,l)} \in D^{\mathcal{I}}$ with $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
- Assume $a_{(k,l)} \in B^{\mathcal{I}}$ with $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
- Assume $a_{(k,l)} \in C^{\mathcal{I}}$ with $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly. $\square$

We now define a mapping $t : \mathbb{N} \times \mathbb{N} \to \mathcal{D}$ as follows. By the axiom 8 in Definition 6, there is $D_i \in \mathcal{D}$ such that $a_{(0,0)} \in {D_i}^{\mathcal{I}}$.

1. $t(0,0) := D_i$ with $a_{(0,0)} \in {D_i}^{\mathcal{I}}$. From the axioms 9, 2, 6 in Definition 6 and Claim 5, there are $D_x^{(0,0)}, D_y^{(0,0)} \in \mathcal{D}$ such that $(D_i, D_x) \in \mathcal{H}$, $(D_i, D_y^{(0,0)}) \in \mathcal{V}$, and $\langle a_{(0,0)}, a_{(1,0)} \rangle \in X^{\mathcal{I}}$ with $a_{(1,0)} \in {D_x^{(0,0)}}^{\mathcal{I}}$, $\langle a_{(0,0)}, a_{(0,1)} \rangle \in Y^{\mathcal{I}}$ with $a_{(0,1)} \in {D_y^{(0,0)}}^{\mathcal{I}}$. Therefore, we define $t(1,0) := D_x^{(0,0)}, t(0,1) := D_y^{(0,0)}$. Since X, Y are functional and D_h are disjoint for all $D_h \in \mathcal{D}$ hence such $D_x^{(0,0)}, D_y^{(0,0)}$ are uniquely determined from D_i.
   Moreover, from the axiom 9, 2, 6 in Definition 6, there are $D_y^{(1,0)}, D_x^{(0,1)} \in \mathcal{D}$ such that $(D_x^{(0,0)}, D_y^{(1,0)}) \in \mathcal{H}$, $(D_y^{(0,0)}, D_x^{(0,1)}) \in \mathcal{V}$, and $\langle a_{(1,0)}, a_{(1,1)} \rangle \in Y^{\mathcal{I}}$ with $a_{(1,1)} \in {D_y^{(1,0)}}^{\mathcal{I}}$, $\langle a_{(0,1)}, a'_{(1,1)} \rangle \in X^{\mathcal{I}}$ with $a'_{(1,1)} \in {D_x^{(0,1)}}^{\mathcal{I}}$. By the axioms 11, 12, 2, 6 in Definition 6 we have $\langle a_{(1,0)}, a_{(1,1)} \rangle \in {Y_2^1}^{\mathcal{I}}$, $\langle a_{(0,1)}, a'_{(1,1)} \rangle \in {X_2^1}^{\mathcal{I}}$. From Claim 5 we have $a_{(1,1)} = a'_{(1,1)}$. This implies that $D_y^{(1,0)} = D_x^{(0,1)}$ since $D_y^{(1,0)}, D_x^{(0,1)}$ are disjoint by the axiom 8 in Definition 6. Therefore we can define $t(1,1) := D_y^{(1,0)} = D_x^{(0,1)}$.
2. Assume that $t(i,j) := D_{i'}$ with $a(i,j) \in {D_{i'}}^{\mathcal{I}}$. From the axiom 9, 2, 6 in Definition 6 and Claim 5, there are $D_x^{(i,j)}, D_y^{(i,j)} \in \mathcal{D}$ such that $(D_x^{(i,j)}, D_y^{(i,j)}) \in \mathcal{H}$, $(D_x^{(i,j)}, D_y^{(i,j)}) \in \mathcal{V}$, and $\langle a_{(i,j)}, a_{(i+1,j)} \rangle \in X^{\mathcal{I}}$ with $a_{(i+1,j)} \in {D_x^{(i,j)}}^{\mathcal{I}}$, $\langle a_{(i,j)}, a_{(i,j+1)} \rangle \in Y^{\mathcal{I}}$ with $a_{(i,j+1)} \in {D_y^{(i,j)}}^{\mathcal{I}}$. Therefore, $t(i+1,j) := D_x^{(i,j)}, t(i,j+1) := D_y^{(i,j)}$. Since X, Y are functional and D_h are disjoint for all $D_h \in \mathcal{D}$ hence such $D_x^{(i,j)}, D_y^{(i,j)}$ are uniquely determined from $D_{i'}$.
   Moreover, from the axiom 9, 2, 6 in Definition 6, there are $D_y^{(i+1,j)}, D_x^{(i,j+1)} \in \mathcal{D}$ such that $(D_x^{(i,j)}, D_y^{(i+1,j)}) \in \mathcal{H}$, $(D_y^{(i,j)}, D_x^{(i,j+1)}) \in \mathcal{V}$, and $\langle a_{(i+1,j)}, a_{(i+1,j+1)} \rangle \in Y^{\mathcal{I}}$ with $a_{(i+1,j+1)} \in {D_y^{(i+1,j)}}^{\mathcal{I}}$, $\langle a_{(i,j+1)}, a'_{(i+1,j+1)} \rangle \in X^{\mathcal{I}}$ with $a'_{(i+1,j+1)} \in {D_x^{(i,j+1)}}^{\mathcal{I}}$. We now distinguish the following cases:
   (a) Assume that $a_{(i,j)} \in A^{\mathcal{I}}$. From Claim 5 and the axiom 8 in Definition 6 we can show $D_y^{(i+1,j)} = D_x^{(i,j+1)}$. Therefore we can define $t(i+1, j+1) := D_y^{(i+1,j)} = D_x^{(i,j+1)}$.
   (b) Assume that $a_{(i,j)} \in B^{\mathcal{I}}$. Similarly.
   (c) Assume that $a_{(i,j)} \in C^{\mathcal{I}}$. Similarly.
   (d) Assume that $a_{(i,j)} \in D^{\mathcal{I}}$. Similarly.

It remains to be shown that (1) t is well defined, (2) the horizontal and vertical matching conditions are satisfied.

(1) is obvious from the construction of the mapping t.

(2) From the definition of t, for each $a_{(k,l)}$ there is a $D_i \in \mathcal{D}$ such that $t(k,l) = D_i$ and $a_{(k,l)} \in {D_i}^{\mathcal{I}}$. Again, by the construction of t, there are $D_j, D_k \in \mathcal{D}$ such that $t(k+1,l) = D_j$, $t(k,l+1) = D_j$ and $a_{(k+1,l)} \in {D_j}^{\mathcal{I}}$, $a_{(k,l+1)} \in {D_k}^{\mathcal{I}}$. By the axioms 2 and 9, we have $\langle D_i, D_j \rangle \in \mathcal{H}$ and $\langle D_i, D_k \rangle \in \mathcal{V}$.