Decidability of Description Logics with Transitive Closure of Roles in Concept and Role Inclusion Axioms

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Abstract

This paper investigates Description Logics which allow transitive closure of roles to occur not only in concept inclusion axioms but also in role inclusion axioms. First, we propose a decision procedure for the description logic $\mathcal{SHIO}_+$, which is obtained from $\mathcal{SHIO}$ by adding transitive closure of roles. Next, we show that $\mathcal{SHIO}_+$ has the *finite model property* by providing a upper bound on the size of models of satisfiable $\mathcal{SHIO}_+$-concepts with respect to sets of concept and role inclusion axioms. Additionally, we prove that if we add number restrictions to $\mathcal{SHI}_+$ then the satisfiability problem is undecidable.

Introduction

The ontology language OWL-DL (Patel-Schneider, Hayes, and Horrocks 2004) is widely used to formalize resources on the Semantic Web. This language is mainly based on the description logic $\mathcal{SHOIN}$ which is known to be decidable (Tobies 2000). Although $\mathcal{SHOIN}$ is expressive and provides *transitive roles* to model transitivity of relations, we can find several applications in which *the transitive closure of roles*, that is more expressive than transitive roles, is necessary. An example in (Sattler 2001) describes two categories of devices as follows: (1) Devices have as their direct part a battery: $\mathsf{Device} \sqcap \exists \mathsf{hasPart}.\mathsf{Battery}$, (2) Devices have at some level of decomposition a battery: $\mathsf{Device} \sqcap \exists \mathsf{hasPart}^+.\mathsf{Battery}$. However, if we now define $\mathsf{hasPart}$ as a *transitive role*, the concept $\mathsf{Device} \sqcap \exists \mathsf{hasPart}.\mathsf{Battery}$ does not represent the devices as described above since it does not allow one to describe these categories of devices as two different sets of devices. We now consider another example in which we need to use the transitive closure of roles in role inclusion axioms.

Example 1 *A process accepts a set* $\mathbf{S}$ *of possible states where* $\mathsf{start} \in \mathbf{S}$ *is the initial state. The process can reach two disjoint phases* $\mathsf{A}, \mathsf{B} \subseteq \mathbf{S}$*, considered as two sets of states. To go from a state to another one, the process has to perform an action* a *or* b*. Sometimes, it can execute a* jump *that implies a sequence of actions* next*.*

To specify the behavior of the process as described, we might need a role name next *to express the fact that a state follows another one, a nominal* o *for* start*, a role name* jump *for jumps, concept names* A, B *for the phases and the following axioms:*

(1) $\mathsf{o} \sqsubseteq \neg\mathsf{A} \sqcap \neg\mathsf{B}$, $\mathsf{A} \sqcap \mathsf{B} \sqsubseteq \bot$, $\mathsf{o} \sqsubseteq \forall \mathsf{next}^-.\bot$

(2) $\top \sqsubseteq \exists \mathsf{next}.\top$, $\mathsf{jump} \sqsubseteq \mathsf{next}^+$

Since jumps are arbitrarily executed over $\mathbf{S}$ *and they form (non-directed) cycles with* next *instances, we cannot use concept axioms to express them. In addition, if a transitive role is used instead of transitive closure, we cannot express the property : an execution of* jump *implies a sequence of actions* next*. Therefore, the axiom* $\mathsf{jump} \sqsubseteq \mathsf{next}^+$ *is necessary.*

Example 2 *We now restrict the behavior of the process in Example 1 by providing two more properties: (i) each state has at most one predecessor and successor state; (ii) the process starts from* start *and if it reaches a state in* B *then it has already got through a state in* A*. In order to take into account the new properties, we need to add the following axioms:*

(3) $\top \sqsubseteq \leq 1\ \mathsf{next}.\top$, $\top \sqsubseteq \leq 1\ \mathsf{next}^-.\top$, $\mathsf{B} \sqsubseteq \exists \mathsf{jump}^-.\mathsf{A}$

Assume that $\mathcal{I}$ *is a model of the nominal* o *w.r.t. the axioms. From the axioms in (1), Example 1, there is a sequence of states* $\mathsf{o}^\mathcal{I}, s_1^\mathcal{I}, \cdots, s_n^\mathcal{I}$ *such that* $\langle \mathsf{o}^\mathcal{I}, s_1^\mathcal{I}\rangle, \cdots, \langle \mathsf{s}_i^\mathcal{I}, s_{i+1}^\mathcal{I}\rangle \in \mathsf{next}^\mathcal{I}$ *for all* $i \in \{1, \cdots, n-1\}$*. If* $s_n^\mathcal{I} \in \mathsf{B}^\mathcal{I}$ *then, by the axioms in (2), Example 1, and the last one in (3) there are* $t_1^\mathcal{I}, \cdots, t_m^\mathcal{I}$ *with* $t_m^\mathcal{I} = s_n^\mathcal{I}$*,* $t_1^\mathcal{I} \in \mathsf{A}^\mathcal{I}$ *such that* $\langle \mathsf{t_i}^\mathcal{I}, t_{i+1}^\mathcal{I}\rangle \in \mathsf{next}^\mathcal{I}$ *for all* $i \in \{1, \cdots, m-1\}$*. Due to the axioms related number restrictions in (3) and* $\mathsf{o} \sqsubseteq \forall \mathsf{next}^-.\bot$*, we have* $m \leq n$ *and* $t_{m-i}^\mathcal{I} = s_{n-i}^\mathcal{I}$ *for all* $i \in \{1, \cdots, m-1\}$*.*

Such examples motivate the study of Description Logics (DL) that allow the transitive closure of roles to occur in both concept and role inclusion axioms. We introduce in this work a DL that can express the process as described in Example 1 and propose a decision procedure for concept satisfiability problem in this DL. In addition, a more expressive DL that can capture the process in Example 2 is also defined. Unfortunately, we show that this DL is undecidable.

To the best of our knowledge, the decidability of $\mathcal{SHIO}_+$, which is obtained from $\mathcal{SHIO}$ by adding transitive closure of roles, is unknown. (Leduc 2009) has es-

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tablished a decision procedure for concept satisfiability in $\mathcal{SHI}_+$ ($\mathcal{SHIO}_+$ without nominal) by using neighborhoods to build completion graphs. In the literature, many decidability results in DLs can be obtained from their counterparts in modal logics ((Giacomo and Lenzerini. 1994), (Giacomo and Lenzerini 1995)). However, these counterparts do not take into account expressive role inclusion axioms. In particular, (Giacomo and Lenzerini 1995) has shown the decidability of a very expressive DL, so-called $\mathcal{CATS}$, including $\mathcal{SHIQ}$ with the transitive closure of roles but not allowing it to occur in role inclusion axioms. (Giacomo and Lenzerini 1995) has pointed out that the complexity of concept subsumption in $\mathcal{CATS}$ is EXPTIME-complete by translating $\mathcal{CATS}$ into the logic Converse PDL in which inference problems are well studied.

Recently, there have been some works in (Horrocks and Sattler 2004) and (Horrocks, Kutz, and Sattler 2006) which have attempted to augment the expressiveness of role inclusion axioms. A decidable logic, namely $\mathcal{SROIQ}$, resulting from these efforts allows for new role constructors such as composition, disjointness and negation. In addition, (Ortiz 2008) has introduced a DL, so-called $\mathcal{ALCQIb}^+_{reg}$, which can capture $\mathcal{SRIQ}$ ($\mathcal{SROIQ}$ without nominal), and obtained the worst-case complexity (EXPTIME-complete) of the satisfiability problem by using automata-based technique. $\mathcal{ALCQIb}^+_{reg}$ allows for a rich set of operators on roles by which one can simulate role inclusion axioms. However, transitive closures in role inclusion axioms are expressible neither in $\mathcal{SROIQ}$ nor in $\mathcal{ALCQIb}^+_{reg}$.

In addition, tableaux-based algorithms for expressive DLs like $\mathcal{SHIQ}$ (Horrocks, Sattler, and Tobies 1999) and $\mathcal{SHOIQ}$ (Horrocks and Sattler 2007) result in efficient implementations. This kind of algorithms relies on two structures, the so-called *tableau* and *completion graph*. Roughly speaking, a tableau for a concept represents a model for the concept and it is possibly infinite. A tableau translates satisfiability of all given concept and role inclusion axioms into the satisfiability of semantic constraints imposed *locally* on each individual of the tableau. This feature of tableaux will be called *local satisfiability property*. In turn, a completion graph for a concept is a *finite* representation from which a tableau can be built. The algorithm in (Baader 1991) for satisfiability in $\mathcal{ALC}_{reg}$ (including the transitive closure of roles and other role operators) introduced a method to deal with loops which can hide unsatisfiable nodes.

Regarding undecidability results, (Horrocks, Kutz, and Sattler 2006) has shown that an arbitrary extension of role inclusion axioms, such as adding $R \circ S \sqsubseteq P$, may lead to undecidability. Additionally, as it turned out by (Horrocks, Sattler, and Tobies 1999), the interaction between transitive roles and number restrictions causes also undecidability. The technique used to prove these undecidability results is to reduce the domino problem, which is known to be undecidable (Berger 1966), to the problem in question.

The contribution of the present paper consists of (i) proving that $\mathcal{SHIO}_+$ has the *finite model property*, and so is decidable by providing an upper bound on the size of models of satisfiable $\mathcal{SHIO}_+$-concepts with respect to (w.r.t.) sets of concept and role inclusion axioms, (iii) establishing a reduction of the domino problem to the concept satisfiability problem in the logic $\mathcal{SHIN}_+$ that is obtained from $\mathcal{SHI}_+$ by adding number restrictions on *simple* roles i.e. roles do not subsume any transitive role. This reduction shows that $\mathcal{SHIN}_+$ is undecidable.

The Description Logic $\mathcal{SHIO}_+$

The logic $\mathcal{SHIO}_+$ is an extension of $\mathcal{SHIO}$ by allowing for transitive closure of roles. In this section, we present the syntax and semantics of $\mathcal{SHIO}_+$. This includes the definitions of inference problems and how they are interrelated. The definitions reuse notation introduced in (Horrocks and Sattler 2007).

Definition 1 *Let* $\mathbf{R}$ *be a non-empty set of* role names. *We denote* $\mathbf{R}_\mathsf{I} = \{P^- \mid P \in \mathbf{R}\}$, $\mathbf{R}_+ = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$.

$*$ *The set of* $\mathcal{SHIO}_+$*-roles is* $\mathbf{R} \cup \mathbf{R}_\mathsf{I} \cup \mathbf{R}_+$. *A* role inclusion axiom *is of the form* $R \sqsubseteq S$ *for two* $\mathcal{SHIO}_+$*-roles* R *and* S. *A* role hierarchy $\mathcal{R}$ *is a finite set of role inclusion axioms.*

$*$ *An interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ *consists of a non-empty set* $\Delta^\mathcal{I}$ (domain) *and a function* $\cdot^\mathcal{I}$ *which maps each role name to a subset of* $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$ *such that, for* $R \in \mathbf{R}'$, $S \in \mathbf{R}_\mathsf{T}$, $Q \in \mathbf{R}' \cup \mathbf{R}_\mathsf{I}$,

$R^{-\mathcal{I}} = \{\langle x, y\rangle \in (\Delta^\mathcal{I})^2 \mid \langle y, x\rangle \in R^\mathcal{I}\}$, *and*

$Q^{+\mathcal{I}} = \bigcup_{n>0} (Q^n)^\mathcal{I}$ *with* $(Q^1)^\mathcal{I} = Q^\mathcal{I}$,

$(Q^n)^\mathcal{I} = \{\langle x, y\rangle \in (\Delta^\mathcal{I})^2 \mid \exists z \in \Delta^\mathcal{I}, \langle x, z\rangle \in (Q^{n-1})^\mathcal{I}, \langle z, y\rangle \in Q^\mathcal{I}\}$.

An interpretation $\mathcal{I}$ *satisfies a role hierarchy* $\mathcal{R}$ *if* $R^\mathcal{I} \subseteq S^\mathcal{I}$ *for each* $R \sqsubseteq S \in \mathcal{R}$. *Such an interpretation is called a* model *of* $\mathcal{R}$, *denoted by* $\mathcal{I} \models \mathcal{R}$.

$*$ *Function* Inv *returns the inverse of a role as follows:*

$$\mathsf{Inv}(R) := \begin{cases} R^- & \text{if } R \in \mathbf{R}, \\ S & \text{if } R = S^- \text{ where } S \in \mathbf{R}, \\ (Q^-)^+ & \text{if } R = Q^+ \text{ where } Q \in \mathbf{R}, \\ Q^+ & \text{if } R = (Q^-)^+ \text{ where } Q \in \mathbf{R} \end{cases}$$

$*$ *A relation* $\underline{\sqsubseteq}$ *is defined as the transitive-reflexive closure of* $\sqsubseteq$ *on* $\mathcal{R} \cup \{\mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$. *We denote* $S \equiv R$ *iff* $R \underline{\sqsubseteq} S$ *and* $S \underline{\sqsubseteq} R$. *We may abuse the notation by saying* $R \underline{\sqsubseteq} S \in \mathcal{R}$.

Notice that we introduce into role hierarchies axioms $Q \sqsubseteq Q^+$ which allows us (i) to propagate $(\forall Q^+.A)$ correctly, and (ii) to take into account the fact that $R \sqsubseteq S$ implies $R^+ \sqsubseteq S^+$.

Definition 2 *Let* $\mathbf{C}' = \mathbf{C} \cup \mathbf{C}_o$ *be a non-empty set of* concept names *where* $\mathbf{C}$ *is a set of normal concept names and* $\mathbf{C}_o$ *is a set of* nominals.

$*$ *The set of* $\mathcal{SHIO}_+$*-concepts is inductively defined as the smallest set containing all* C *in* $\mathbf{C}'$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$ *where* C *and* D *are* $\mathcal{SHIO}_+$*-concepts,* R *is an* $\mathcal{SHIO}_+$*-role,* S *is a simple role and* $n \in \mathbb{N}$. *We denote* $\bot$ *for* $\neg\top$.

$*$ *An interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ *consists of a non-empty set* $\Delta^\mathcal{I}$ (domain) *and a function* $\cdot^\mathcal{I}$ *which maps each concept name to a subset of* $\Delta^\mathcal{I}$ *such that* $\mathsf{card}\{o^\mathcal{I}\} = 1$ *for all* $o \in \mathbf{C}_o$ *where* $\mathsf{card}\{\cdot\}$ *is denoted for the cardinality of a set* $\{\cdot\}$,

$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$,
$(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}, \langle x, y \rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$,
$(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}}, \langle x, y \rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$
$*$ *$C \sqsubseteq D$ is called a general concept inclusion (GCI) where C, D are $\mathcal{SHIO}_+$-concepts (possibly complex), and a finite set of GCIs is called a terminology $\mathcal{T}$. An interpretation $\mathcal{I}$ satisfies a GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $\mathcal{I}$ satisfies a terminology $\mathcal{T}$ if $\mathcal{I}$ satisfies each GCI in $\mathcal{T}$. Such an interpretation is called a* model *of $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$.*

$*$ *A concept C is called* satisfiable *w.r.t. a role hierarchy $\mathcal{R}$ and a terminology $\mathcal{T}$ iff there is some interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a* model *of C w.r.t. $\mathcal{R}$ and $\mathcal{T}$. A pair $(\mathcal{T}, \mathcal{R})$ is called an $\mathcal{SHIO}_+$* ontology *and said to be consistent if there is a model of $(\mathcal{T}, \mathcal{R})$.*

$*$ *A concept D* subsumes *a concept C w.r.t. $\mathcal{R}$ and $\mathcal{T}$, denoted by $C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in each model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{R})$.*

Notice that a transitive role S (i.e. $\langle x, y \rangle \in S^{\mathcal{I}}, \langle y, z \rangle \in S^{\mathcal{I}}$ implies $\langle x, z \rangle \in S^{\mathcal{I}}$ where $\mathcal{I}$ is an interpretation) can be expressed by using a role axiom $S^+ \sqsubseteq S$. Since negation is allowed in the logic $\mathcal{SHIO}_+$, unsatisfiability and subsumption w.r.t. $(\mathcal{T}, \mathcal{R})$ can be reduced each other: $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable. In addition, we can reduce ontology consistency to concept satisfiability w.r.t. an ontology: $(\mathcal{T}, \mathcal{R})$ is consistent if $A \sqcup \neg A$ is satisfiable w.r.t. $(\mathcal{T}, \mathcal{R})$ for some concept name A.

For the ease of construction, we assume all concepts to be in *negation normal form* (NNF) i.e. negation occurs only in front of concept names. Any $\mathcal{SHIO}_+$-concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in (Horrocks, Sattler, and Tobies 1999). For a concept C, we denote the nnf of C by $\mathsf{nnf}(C)$ and the nnf of $\neg C$ by $\dot\neg C$.

Let D be an $\mathcal{SHIO}_+$-concept in NNF. We define $\mathsf{sub}(D)$ to be the smallest set that contains all sub-concepts of D including D. For an ontology $(\mathcal{T}, \mathcal{R})$, we define the set of all sub-concepts $\mathsf{sub}(\mathcal{T}, \mathcal{R})$ as follows:

$$\begin{array}{lll} \mathsf{sub}(\mathcal{T}, \mathcal{R}) & := & \bigcup\limits_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(\mathsf{nnf}(\neg C \sqcup D), \mathcal{R}) \\ \mathsf{sub}(E, \mathcal{R}) & := & \mathsf{sub}(E) \cup \{\dot\neg C \mid \neg C \in \mathsf{sub}(E)\} \cup \\ & & \{\forall S.C \mid (\forall R.C \in \mathsf{sub}(E), S \underline{\circledast} R) \vee \\ & & (\dot\neg \forall R.C \in \mathsf{sub}(E), S \underline{\circledast} R) \\ & & \text{and } S \text{ occurs in } \mathcal{T} \text{ or } \mathcal{R}\} \end{array}$$

For the sake of simplicity, for each concept D w.r.t. $(\mathcal{T}, \mathcal{R})$ we denote $\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)$ for $\mathsf{sub}(\mathcal{T}, \mathcal{R}) \cup \mathsf{sub}(D)$, and $\mathbf{R}_{(\mathcal{T}, \mathcal{R}, D)}$ for the set of roles occurring in $\mathcal{T}, \mathcal{R}$, D, their inverse and transitive closure. If it is clear from the context we will use $\mathbf{R}$ instead of $\mathbf{R}_{(\mathcal{T}, \mathcal{R}, D)}$.

A decision procedure for $\mathcal{SHIO}_+$

In this section, we establish decidability of $\mathcal{SHIO}_+$ by devising a terminating, sound and complete algorithm for checking the satisfiability of $\mathcal{SHIO}_+$ concepts w.r.t. a terminology and role hierarchy.

In our approach, we define a sub-structure of graphs, called *neighborhood*, which consists of a node together with its neighbors. Such a neighborhood captures all semantic constraints imposed by the logic constructors of $\mathcal{SHIO}$. A graph obtained by "tiling" neighborhoods together allows us to represent in some way a model for a concept in $\mathcal{SHIO}_+$. In fact, we embed in this graph another structure, called *cyclic path*, to express transitive closure of roles. Since all expansion rules for $\mathcal{SHIO}$ can be translated into construction of neighborhoods, the algorithm presented in this paper focuses on defining cyclic paths over such a graph. By this way, the non-determinism resulting from satisfying the transitive closure of roles can be translated into the search in a space of all possible graphs obtained from tiling neighborhoods.

Neighborhood for $\mathcal{SHIO}_+$

Tableau-based algorithms, as presented in (Horrocks and Sattler 2007), use expansion rules representing tableau properties to build a completion graph. Applying expansion rules makes all nodes of a completion graph satisfy semantic constraints imposed by concept definitions in the label associated with each node. This means that *local* satisfiability in such completion graphs is sufficient to ensure *global* satisfiability. The notion of *neighborhood* introduced in Definition 3 expresses exactly the expansion rules for $\mathcal{SHIO}$, consequently, guarantees local satisfiability. Therefore, a completion graph built by a tableau-based algorithm can be considered as set of neighborhoods which are tiled together. In other terms, building a completion tree by applying expansion rules is equivalent to the search of a tiling of neighborhoods.

Definition 3 (Neighborhood) *Let D be an $\mathcal{SHIO}_+$ concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $\mathbf{R}$ be the set of roles occurring in D and $\mathcal{T}, \mathcal{R}$ together with their inverse. A* neighborhood*, denoted (v_B, N_B, l), for D w.r.t. $(\mathcal{T}, \mathcal{R})$ is formed from a* core *node v_B, a set of* neighbor *nodes N_B, edges $\langle v_B, v \rangle$ with $v \in N_B$ and a labelling function l such that $l(u) \in 2^{\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)}$ with $u \in \{v_B\} \cup N_B$ and $l\langle v_B, v \rangle \in 2^{\mathbf{R}}$ with $v \in N_B$.*

1. *A node $v \in \{v_B\} \cup N_B$ is* nominal *if there is $o \in \mathbf{C}_o$ such that $o \in l(v)$. Otherwise, v is a* non-nominal *node;*
2. *A node $v \in \{v_B\} \cup N_B$ is* valid *w.r.t. D and $(\mathcal{T}, \mathcal{R})$ iff*
   (a) *$tbox$-**rule:** If $C \sqsubseteq D \in \mathcal{T}$ then $\mathsf{nnf}(\neg C \sqcup D) \in l(v)$, and*
   (b) *$clash$-**rule:** $\{A, \neg A\} \nsubseteq l(v)$ with any concept name A, and*
   (c) *$\sqcap$-**rule:** If $C_1 \sqcap C_2 \in l(v)$ then $\{C_1, C_2\} \subseteq l(v)$, and*
   (d) *$\sqcup$-**rule:** If $C_1 \sqcup C_2 \in l(v)$ then $\{C_1, C_2\} \cap l(v) \neq \emptyset$.*
3. *A neighborhood $B = (v_B, N_B, l)$ is* valid *iff all nodes $\{v_B\} \cup N_B$ are valid and the following conditions are satisfied:*
   (a) *$\exists$-**rule:** If $\exists R.C \in l(v_B)$ then there is a neighbor $v \in N_B$ such that $C \in l_B(v)$ and $R \in l\langle v_B, v \rangle$;*
   (b) *$rbox$-**rule:** For each $v \in N_B$, if $R \in l\langle v_B, v \rangle$ and $R \underline{\circledast} S$ then $S \in l\langle v_B, v \rangle$;*

(c) *$\forall$-**rule:** For each $v \in N_B$, if $R \in l\langle v_B, v\rangle$ (resp. $R \in \mathsf{Inv}(l\langle v_B, v\rangle)$) and $\forall R.C \in l(v_B)$ (resp. $\forall R.C \in l(v)$) then $C \in l(v)$ (resp. $C \in l(v_B)$);*

(d) *$\forall^+$-**rule:** For each $v \in N_B$, if $Q^+ \in l\langle v_B, v\rangle$ (resp. $Q^+ \in \mathsf{Inv}(l\langle v_B, v\rangle)$), $Q^+ \circledast R \in \mathcal{R}$ and $\forall R.D \in l(v_B)$ (resp. $\forall \mathsf{Inv}(R).D \in l(v)$) then $\forall Q^+.D \in l(v)$ (resp. $\forall \mathsf{Inv}(Q^+).D \in l(v_B)$);*

(e) ***o-rule:** For each $o \in \mathbf{C}_o$, if $o \in l(u) \cap l(v)$ with $\{u, v\} \subseteq \{v_B\} \cup N_B$ then $l(u) = l(v)$;*

(f) *$\leq_\infty$-**rule:** There is at most one node $v \in N_B$ such that $l(v) = \mathscr{C}$ and $l\langle v_B, v\rangle = \mathscr{R}$ for each $\mathscr{C} \in 2^{\mathsf{sub}(\mathcal{T},\mathcal{R},D)}$, $\mathscr{R} \in 2^{\mathbf{R}}$.*

We denote $\mathbb{B}_{(\mathcal{T},\mathcal{R},D)}$ for a set of all valid neighborhoods for D w.r.t. $(\mathcal{T}, \mathcal{R})$. When it is clear from the context we will use $\mathbb{B}$ instead of $\mathbb{B}_{(\mathcal{T},\mathcal{R},D)}$.

The condition 3f in Definition 3 ensures that any neighborhood has a finite number of neighbors.

As mentioned, a valid neighborhood as presented in Definition 3 satisfies all concept definitions in the label associated with the core node. For this reason, neighborhoods can be still used to tile a completion tree for $\mathcal{SHIO}_+$ without taking care of expansion rules for $\mathcal{SHIO}$.

Lemma 1 *Let D be an $\mathcal{SHIO}_+$ concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $(v_B, N_B, l), (v_{B'}, N_{B'}, l)$ be two valid neighborhoods with $l(v_B) = l(v_{B'})$. If there is $v \in N_B$ such that there does not exist any $v' \in N_{B'}$ satisfying $l(v') = l(v)$ and $l\langle v_B, v\rangle = l\langle v_{B'}, v'\rangle$ then the neighborhood $(v_{B'}, N_{B'} \cup \{u\}, l)$ is valid where $l(u) = l(v)$ and $l\langle v_{B'}, u\rangle = l\langle v_B, v\rangle$.*

This lemma holds due to the facts that (i) a valid neighbor in a valid neighborhood B is also a valid neighbor in another valid neighborhood B' if the labels of two core nodes of B and B' are identical, (ii) since $\mathcal{SHIO}_+$ does not allow for number restrictions hence Definition 3 has no restriction on the number of neighbors of a core node.

Completion Tree with Cyclic Paths

As discussed in works related to tableau-based technique, the blocking technique fails in treating DLs with the transitive closure of roles. It works correctly only if the satisfiability of a node in completion tree can be decided from its neighbors and itself i.e. *local* satisfiability must be sufficient for such completion trees. However, the presence of the transitive closure of roles makes satisfiability of a node depend on further nodes which can be arbitrarily far from it.

More precisely, satisfying the transitive closure P^+ in an edge $\langle x, y\rangle$ (i.e. $P^+ \in L\langle x, y\rangle$) is related to a set of nodes on a path rather than a node with its neighbors i.e. it imposes a semantic constraint on a set of nodes $x, x_1, \cdots, x_n, y$ such that they are connected together by P-edges. In general, satisfying the transitive closure is quite nondeterministic since the semantic constraint can lead to be applied to an *arbitrary* number of nodes. In addition, the presence of transitive closure of roles in a role hierarchy makes this difficulty worse. For instance, if $P \sqsubseteq Q^+, Q \sqsubseteq S^+$ are axioms in a role hierarchy then each Q-edge generated for satisfying Q^+ may lead to generate an arbitrary number of S-edges for satisfying S^+.

The most common way for dealing with a new logic constructor is to add a new expansion rule for satisfying the semantic constraint imposed by the new constructor. Such an expansion rule for the transitive closure of roles must:

1. find or create a set of P-edges forming a path for each occurrence of P^+ in the label of edges;
2. deal with non-deterministic behaviours of the expansion rule resulting from the semantics of the transitive closure of roles;
3. enable to control the expansion of completion trees by a new blocking technique which has to take into account the fact that satisfying the transitive closure of a role may add an arbitrary number of new transitive closures to be satisfied.

To avoid these difficulties, our approach does not aim to directly extend the construction of completion trees by using a new expansion rule, but to translate this construction into selecting a "good" completion tree, namely *completion tree with cyclic paths*, from a finite set of trees without taking into account the semantic constraint imposed by the transitive closure of roles. The process of selecting a "good" completion tree is guided by finding in a completion tree (which is well built in advance) a *cyclic path* for each occurrence of the transitive closure of a role.

Summing up, a completion tree with cyclic paths will be built in two stages. The first one which yields a tree consists of tiling valid neighborhoods together such that two neighborhoods are tiled if they have *compatible* neighbors. The second stage deals with the transitive closure of roles by defining cyclic paths over the tree obtained from the first stage. Both of them are presented in Definition 4.

Definition 4 (Completion Tree with Cyclic Paths) *Let D be a $\mathcal{SHIO}_+$ concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $\mathbb{B}$ be the set of all valid neighborhoods for D w.r.t. $(\mathcal{T}, \mathcal{R})$. A tree $\mathbf{T} = (V, E, L)$ for D w.r.t. $(\mathcal{T}, \mathcal{R})$ is defined from $\mathbb{B}_{(\mathcal{T},\mathcal{R},D)}$ as follows.*

1. *If there is a valid neighborhood $(v_0, N_0, l) \in \mathbb{B}$ with $D \in l(v_0)$ then a root node x_0 and successors x of x_0 are added to V such that $L(x_0) = l(v_0)$, and $L(x) = l(v)$, $L\langle x_0, x\rangle = l\langle v_0, v\rangle$ for each $v \in N_0$.*
2. *For each node $x \in V$ with its predecessor x',*
   (a) *If there is an ancestor y of x such that $L(y) = L(x)$ then x is* blocked *by y. In this case, x is a leaf node;*
   (b) *Otherwise, if we find a valid neighborhood (v_B, N_B, l) from $\mathbb{B}$ such that*
      i. *$l(v_B) = L(x), l(v) = L(x')$, $\mathsf{Inv}(\ l\langle v_B, v\rangle\ ) = L\langle x', x\rangle$ for some $v \in N_B$, and*
      ii. *if there is some nominal $o \in \mathbf{C}_o$ such that $o \in l(u) \cap L(w)$ with $u \in N_B \setminus \{v\}$, $w \in V$ then $l(u) = L(w)$*

      then we add a successor y of x for each $u \in N_B \setminus \{v\}$ such that $L(y) = l(u)$ and $L(\langle x, y\rangle) = l(\langle v_B, u\rangle)$.

We say a node x is a R-successor of $x' \in V$ if $R \in L\langle x', x\rangle$. A node x is called a R-neighbor of x' if x is a R-successor

of x' or x' is a $\mathsf{Inv}(R)$-successor of x. In addition, a node x is called a R-block of x' if x blocks a R-successor of x' or x' blocks a $\mathsf{Inv}(R)$-successor of x.
$\mathbf{T} = (V, E, L)$ *is called a* completion tree with cyclic paths *if for each $\langle u, v\rangle \in E$ such that $Q^+ \in L\langle u, v\rangle$ and $Q \notin L\langle u, v\rangle$ there exists a* cyclic path $\varphi = \langle x_0, \cdots, x_n\rangle$ *which is formed from nodes $v_i \in V$ and satisfies the following conditions:*

- *$x_0 = u$ and x_i is not blocked for all $i \in \{0, \cdots, n\}$;*
- *There do not exist $i, j \in \{1, \cdots, n-1\}$ with $j > i$ such that $L(x_i) = L(x_j)$;*
- *$L(x_n) = L(v)$ and x_i is a Q-neighbor or Q-block of x_{i+1} for all $0 \le i \le n-1$.*

In this case, φ is called a cyclic path and denoted by $\varphi_{\langle u,v\rangle}$.

Note that the construction of completion trees uses the equality blocking $L(x) = L(y)$ for termination condition. A completion tree encapsulates the following notions: neighborhood, blocking condition and cyclic path. The first one captures the semantics of all logic constructors except for the transitive closure of roles. The second one which was introduced in (Horrocks, Sattler, and Tobies 1999) is crucial for obtaining a finite representation of a possibly infinite model. The third one represents the transitive closure of roles.

At this point we have gathered all necessary elements to introduce a decision procedure for the concept satisfiability in $\mathcal{SHIO}_+$. However, in order to provide a upper bound on the size of models of satisfiable $\mathcal{SHIO}_+$-concepts we need an extra structure, namely *reduced tableau*.

Definition 5 (Reduced Tableau) *Let $\mathbf{T} = (V, E, L)$ be a completion tree with cyclic paths for a $\mathcal{SHIO}_+$-concept D with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. An equivalence relation $\sim$ over V is defined as follows: $x \sim y$ iff $L(x) = L(y)$.*

Let $V/\sim := \{[x] \mid x \in V\}$ be the set of all equivalence classes of V by $\sim$. A graph $G = (V/\sim, E', L)$ is called reduced tableau *for D w.r.t. $(\mathcal{T}, \mathcal{R})$ if:*

- *$L([x]) = L(x')$ for any $x' \in [x]$;*
- *$\langle [x], [y]\rangle \in E'$ iff there are $x' \in [x], y' \in [y]$ such that $\langle x', y'\rangle \in E$;*
- $$L(\langle [x], [y]\rangle) = \bigcup_{x'\in[x],y'\in[y],\langle x',y'\rangle\in E} L(\langle x', y'\rangle) \cup \bigcup_{x'\in[x],y'\in[y],\langle y',x'\rangle\in E} \mathsf{Inv}(L\langle y', x'\rangle)$$
  where $\mathsf{Inv}(L\langle x, y\rangle) = \{\mathsf{Inv}(R) \mid R \in L\langle y, x\rangle\}$

A reduced tableau as defined in Definition 5 identifies nodes whose labels are the same. This construction preserves not only the validity of neighborhoods but also cyclic paths. Indeed, what may be locally changed is the number of neighbors of a node from completion tree. Again, since number restrictions are not allowed in $\mathcal{SHIO}_+$ this change does not violate the validity of neighborhoods. Moreover, a node that is a R-neighbor of another one remains to be a R-neighbor after identifying nodes whose labels are the same. This explains why cyclic paths are preserved.

Lemma 2 *Let D be a $\mathcal{SHIO}_+$-concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $G = (V/\sim, E', L)$ be a reduced tableau for D w.r.t. $(\mathcal{T}, \mathcal{R})$. We define $\Delta^{\mathcal{I}} = V/\sim$ and a function $\cdot^{\mathcal{I}}$ that maps:*

- *each concept name A occurring in $D, \mathcal{T}$ and $\mathcal{R}$ to $A^{\mathcal{I}} \subseteq V/\sim$ such that $A^{\mathcal{I}} = \{[x] \mid A \in L([x])\}$;*
- *each role name R occurring in $D, \mathcal{T}$ and $\mathcal{R}$ to $R^{\mathcal{I}} \subseteq (V/\sim)^2$ such that $R^{\mathcal{I}} = \{\langle [x], [y]\rangle \mid R \in L\langle [x], [y]\rangle\} \cup \{\langle [y], [x]\rangle \mid \mathsf{Inv}(R) \in L\langle [x], [y]\rangle\}$*

If D has a reduced tableau G then $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a model of D w.r.t. $(\mathcal{T}, \mathcal{R})$.

The following lemma affirms that a reduced tableau of a concept D can represent a model of this concept. A mentioned above, a reduced tableau preserves the validity of neighborhoods and cyclic paths of a completion tree. According to Definition 3, all semantic constraints imposed by concept definitions in the label of a node are satisfied. Moreover, each cyclic path represents a sequence of nodes that allows to satisfy the transitive closure of a role P^+ occurring in the label of an edge. These properties help prove Lemma 2.

The following proposition is an immediate consequence of Lemma 2.

Proposition 1 *Let D be a $\mathcal{SHIO}_+$-concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. If there is a completion tree with cyclic paths $\mathbf{T}$ for D w.r.t. $(\mathcal{T}, \mathcal{R})$ then D has a finite model whose size is bounded by an exponential function in the size of $D, \mathcal{T}$ and $\mathcal{R}$.*

Indeed, by the construction of the reduced tableau $G = (V/\sim, E', L)$, the number of nodes of G is bounded by 2^K where K is the cardinality of $\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)$, which is a polynomial function in the size of $D, \mathcal{T}$ and $\mathcal{R}$.

Lemma 3 *Let D be a $\mathcal{SHIO}_+$-concept. Let $\mathcal{T}$ and $\mathcal{R}$ be a terminology and role hierarchy. If D has a model w.r.t. $(\mathcal{T}, \mathcal{R})$ then there exists a completion tree with cyclic paths.*

A proof of Lemma 3 can be performed in three steps. First, we define directly valid neighborhoods from individuals of a model. Next, a completion tree can be built by tiling valid neighborhoods with help of role relationships between individuals of the model. Finally, cyclic paths are embedded into the obtained tree by devising paths from finite cycles for the transitive closure of roles in the model. Lemma 1 makes possible adding a new node to a given neighborhood as neighbor if the new node is a neighbor of a node whose label equals to that of the core node of the neighborhood.

From the construction of completion trees with cyclic paths according to Definition 4 and Lemma 2 and 3, we can devise immediately Algorithm 1 for the concept satisfiability in $\mathcal{SHIO}_+$.

Lemma 4 (Termination) *Algorithm 1 for $\mathcal{SHIO}_+$ terminates and the size of completion trees is bounded by a double exponential function in the size of inputs.*

Termination of Algorithm 1 is a consequence of the following facts: (i) the number of valid neighborhoods is bounded, (ii) the size of completion trees which are tiled

Input : Concept D, terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$
Output: IsSatisfiable(D)

foreach *Tree* $\mathbf{T} = (V, E, L)$ *obtained from tiling valid neigborhoods* **do**
    if *For each* $\langle x, y\rangle \in E$ *with* $Q^+ \in L\langle x, y\rangle, Q \notin L\langle x, y\rangle$, $\mathbf{T}$ *has a* $\varphi_{\langle x,y\rangle}$ **then**
        return `true`;
return `false`;

Algorithm 1: Decision procedure for concept satisfiability in $\mathcal{SHIO}_+$

from valid neighborhoods is bounded by $(2^{m\times n})^{2^{n\times(m+1)}}$ where $m = \mathsf{card}\{\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)\}, n = \mathsf{card}\{\mathbf{R}\}$.

Algorithm 1 is highly complex since it is not a goal-directed procedure. Such an exhaustive behavior is very different from that of tableau-based algorithms in which the construction of a completion tree is inherited from step to step. In Algorithm 1, when a tree obtained from tiling neighborhoods cannot satisfy an occurrence of the transitive closure of a role (after satisfying others), the construction of tree has to restart. The following theorem is a direct consequence of Lemma 3 and 4.

Theorem 1 *Algorithm 1 is a decision procedure for the satisfiability of $\mathcal{SHIO}_+$-concepts w.r.t. a terminology and role hierarchy, it runs in deterministic 3-*EXPTIME *and nondeterministic 2-*EXPTIME.

Thm. 1 is a consequence of the following facts: (i) the size of completion trees is bounded by a double exponential function in the size of inputs , and (ii) the number of of completion trees is bounded by a triple exponential function in the size of inputs.

Remark 1 *From the construction of reduced tableaux in Definition 5, we can devise an algorithm for deciding the satisfiability in $\mathcal{SHIO}_+$ which runs in* NEXPTIME. *In fact, such an algorithm can check the validity of neighborhoods and cycles for transitive closures in a graph whose size is bounded by an exponential function in the size of the input.*

Adding number restrictions to $\mathcal{SHI}_+$

The logic $\mathcal{SHIN}_+$ is obtained from $\mathcal{SHI}_+$ ($\mathcal{SHIO}_+$ without nominals) by allowing, additionally, for number restrictions, i.e., for concepts of the form $(\geq n\, R)$ and $(\leq n\, R)$ where R is a *simple* role and n is a non-negative integer.

Definition 6 *Let $\mathbf{R}, \mathbf{C}$ be sets of role and concept names. The set of $\mathcal{SHIN}_+$-roles, role hierarchy $\mathcal{R}$ and model $\mathcal{I}$ of $\mathcal{R}$ are defined similarly to those in Def. 1.*
$*$ *A role R is called* simple *w.r.t. $\mathcal{R}$ iff $(Q^+ \sqsubseteq^* R) \notin \mathcal{R}$ for any $Q^+ \in \mathbf{R}_+$.*
$*$ *The set of $\mathcal{SHIN}_+$-concepts is inductively defined as the smallest set containing all $C \in \mathbf{C}$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n\, S)$ and $(\geq n\, S)$ where C and D are $\mathcal{SHIN}_+$-concepts, R is a $\mathcal{SHIN}_+$-role and S is a simple role. We denote $\bot$ for $\neg\top$.*
$*$ *An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non empty set $\Delta^{\mathcal{I}}$* (domain) *and a function $\cdot^{\mathcal{I}}$ which maps each concept name to a subset of $\Delta^{\mathcal{I}}$. In addition, the function $\cdot^{\mathcal{I}}$ satisfies the conditions for the logic constructors in $\mathcal{SHI}_+$ (as introduced in Def. 2 without nominal), and*
$(\geq n\, R)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \mathsf{card}\{y \in \Delta^{\mathcal{I}} \mid \langle x, y\rangle \in R^{\mathcal{I}}\} \geq n\}$,
$(\leq n\, R)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \mathsf{card}\{y \in \Delta^{\mathcal{I}} \mid \langle x, y\rangle \in R^{\mathcal{I}})\} \leq n\}$
$*$ *Satisfiability of a $\mathcal{SHIN}_+$-concept C w.r.t. a role hierarchy $\mathcal{R}$ and a terminology $\mathcal{T}$ is defined similarly to that in Def. 2.*

A definition for tableaux in $\mathcal{SHIN}_+$ would be given by combining the properties from tableaux in $\mathcal{SHIN}$ and $\mathcal{SHI}_+$. In addition, a definition for neighborhoods in $\mathcal{SHIN}_+$ would be provided if we adopt that there may be two neighborhoods such that the labels of their core nodes are identical but they cannot be merged together i.e. a property being similar to Lem. 1 no longer holds for $\mathcal{SHIN}_+$. In such a situation, the local information related to the labels of the ending nodes of a path would be not sufficient to form a cycle. This prevents us from embedding cyclic paths to a normalization trees in guaranteeing the soundness and completeness. Note that for the logics $\mathcal{SHI}_+$ and $\mathcal{SHIO}_+$ we can transform a reduced tableau to a tableau such that if any two nodes x, y having the same label then there is an isomorphism between the two neighborhoods (x, N_x, l) and (y, N_y, l). This means that if we know the label of a node in such a tableau it is possible to determine all nodes which are arbitrarily far from this node. This property does not hold for $\mathcal{SHIN}_+$ tableaux.

Notice that this difficulty does not occur when embedding cyclic paths to a normalization tree is not necessary. Therefore, it is very believable that a decision procedure for $\mathcal{SHIQ}$ with the transitive closure of roles appearing only in concept inclusion axioms could be obtained by tiling neighborhoods for $\mathcal{SHIQ}$.

In the sequel, we show that the difficulty mentioned is insurmountable i.e. the concept satisfiability problem in $\mathcal{SHIN}_+$ is undecidable. The undecidability proof uses a reduction of the domino problem (Berger 1966). The following definition, which is taken from (Horrocks, Sattler, and Tobies 1999), reformulates the problem in a more precise way.

Definition 7 *A domino system $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ consists of a non-empty set of domino types $\mathcal{D} = \{D_1, \cdots, D_l\}$ and of sets of horizontally and vertically matching pairs $\mathcal{H} \subseteq \mathcal{D}\times\mathcal{D}$ and $\mathcal{V} \subseteq \mathcal{D}\times\mathcal{D}$. The problem is to determine if, for a given $\mathbf{D}$, there exists a tiling of an $\mathbb{N}\times\mathbb{N}$ grid such that each point of the grid is covered with a domino type in $\mathcal{D}$ and all horizontally and vertically adjacent pairs of domino types are in $\mathcal{H}$ and $\mathcal{V}$ respectively, i.e., a mapping $t : \mathbb{N}\times\mathbb{N} \to \mathcal{D}$ such that for all $m, n \in \mathbb{N}$, $\langle t(m, n), t(m+1, n)\rangle \in \mathcal{H}$ and $\langle t(m, n), t(m, n+1)\rangle \in \mathcal{V}$.*

The reduction of the domino problem to the satisfiability of $\mathcal{SHIN}_+$-concepts will be carried out by (i) constructing a concept, namely A, and two sets of concept and role inclusion axioms, namely $\mathcal{T}_D$ and $\mathcal{R}_D$, and (ii) showing that the

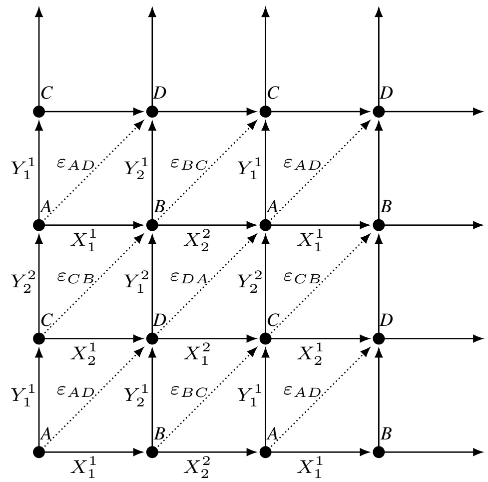

Figure 1: The grid illustrates a model of the concept A w.r.t the axioms

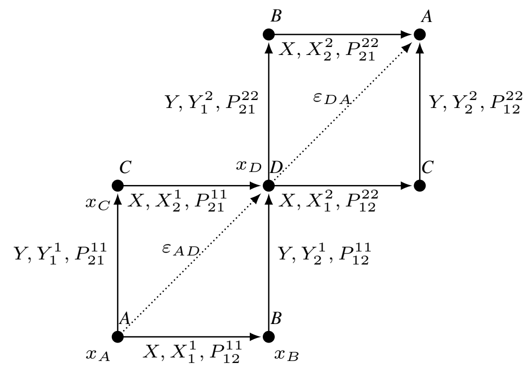

Figure 2: How each square can be formed from a diagonal represented by an ε

domino problem is equivalent to the satisfiability of A w.r.t. $\mathcal{T}_D$ and $\mathcal{R}_D$. Axioms in Definition 8 specify a grid (Fig.1) that represents such a domino system.

Globally, given a domino set $\mathcal{D} = \{D_1, \cdots, D_l\}$, we need axioms that impose that each point of the plane is covered by exactly one $D_i^{\mathcal{I}}$ (axiom 8 in Definition. 8) and ensure that each D_i is compatibly placed in the horizontal and vertical lines (axiom 9). Locally, the key idea is to use $\mathcal{SHIN}_+$ axioms for describing the grid as illustrated in Figure 2. For example, we consider how a square of the grid can be formed. Axiom 10 in Definition 8 says that if A has an instance $x_A^{\mathcal{I}}$ with an interpretation $\mathcal{I}$, then there are three instances $x_B^{\mathcal{I}}, x_C^{\mathcal{I}}, x_D^{\mathcal{I}}$ in $B^{\mathcal{I}}, C^{\mathcal{I}}, D^{\mathcal{I}}$, respectively, such that $\langle x_A^{\mathcal{I}}, x_B^{\mathcal{I}}\rangle \in {X_1^1}^{\mathcal{I}}$, $\langle x_A^{\mathcal{I}}, x_C^{\mathcal{I}}\rangle \in {Y_1^1}^{\mathcal{I}}$ and $\langle x_A^{\mathcal{I}}, x_D^{\mathcal{I}}\rangle \in {\varepsilon_{AD}}^{\mathcal{I}}$. These instances are distinct since A, B, C, D are disjoint by axioms 10, 11, 12 and 13. In addition, by axioms 11, 12, there are $x_D'^{\mathcal{I}}, x_D''^{\mathcal{I}} \in D^{\mathcal{I}}$ such that $\langle x_B^{\mathcal{I}}, x_D'^{\mathcal{I}}\rangle \in {Y_2^1}^{\mathcal{I}}$, $\langle x_C^{\mathcal{I}}, x_D''^{\mathcal{I}}\rangle \in {X_2^1}^{\mathcal{I}}$. This is depicted in Figure 2.

Since P_{12}^{11} subsumes X_1^1, Y_2^1 by axiom 1, we have $\langle x_A^{\mathcal{I}}, x_D'^{\mathcal{I}}\rangle \in ({P_{12}^{11}}^+)^{\mathcal{I}}$. Moreover, since P_{12}^{11} is functional by axiom 5, $\langle x_A^{\mathcal{I}}, x_D^{\mathcal{I}}\rangle \in ({P_{12}^{11}}^+)^{\mathcal{I}}$ by axiom 3, and ${\varepsilon_{AD}}^{\mathcal{I}} \subseteq ({P_{12}^{11}}^+)^{\mathcal{I}}$ by axiom 3, there are two possibilities: (i) $x_D^{\mathcal{I}} = x_D'^{\mathcal{I}}$, and (ii) there is $y^{\mathcal{I}}$ such that $\langle x_D'^{\mathcal{I}}, y^{\mathcal{I}}\rangle \in (P_{12}^{11})^{\mathcal{I}}$. This contradicts axiom 13. Therefore, $x_D^{\mathcal{I}} = x_D'^{\mathcal{I}}$. Similarly, we can get $x_D^{\mathcal{I}} = x_D''^{\mathcal{I}}$. By this way, the axioms in Definition 8 can yield a grid as in Figure 2 if concept A is satisfiable.

Definition 8 *Let* $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ *be a domino system with* $\mathcal{D} = \{D_1, \cdots, D_l\}$. *Let* N_C *and* N_R *be sets of concept and role names such that* $N_C = \{A, B, C, D\} \cup \mathcal{D}$, $N_R = \{X_j^i \mid i, j \in \{1,2\}\} \cup \{X, Y\} \cup \{P_{rs}^{ij} \mid i, j, r, s \in \{1,2\}, r \neq s\} \cup \{\varepsilon_{AD}, \varepsilon_{DA}, \varepsilon_{BC}, \varepsilon_{CB}\}$

Role hierarchy:

1. $X_r^i \sqsubseteq P_{rs}^{ij}, Y_s^j \sqsubseteq P_{rs}^{ij}$ *for all* $i, j, r, s \in \{1,2\}, r \neq s$,
2. $X_r^i \sqsubseteq X, Y_r^i \sqsubseteq Y$ *for all* $i, r \in \{1,2\}$,
3. $\varepsilon_{AD} \sqsubseteq {P_{12}^{11}}^+, \varepsilon_{AD} \sqsubseteq {P_{21}^{11}}^+, \varepsilon_{DA} \sqsubseteq {P_{12}^{22}}^+, \varepsilon_{DA} \sqsubseteq {P_{21}^{22}}^+$,
4. $\varepsilon_{BC} \sqsubseteq {P_{21}^{21}}^+, \varepsilon_{BC} \sqsubseteq {P_{12}^{21}}^+$, $\varepsilon_{CB} \sqsubseteq {P_{21}^{12}}^+, \varepsilon_{CB} \sqsubseteq {P_{12}^{12}}^+$,
   Concept inclusion axioms:
5. $\top \sqsubseteq \leq 1 P_{rs}^{ij}$ *for all* $i, j, r, s \in \{1,2\}, r \neq s$,
6. $\top \sqsubseteq\ \leq 1X, \top \sqsubseteq\ \leq 1Y$,
7. $\top \sqsubseteq \leq 1\varepsilon_{AD}$, $\top \sqsubseteq \leq 1\varepsilon_{DA}$, $\top \sqsubseteq \leq 1\varepsilon_{BC}$, $\top \sqsubseteq \leq 1\varepsilon_{CB}$,
8. $\top \sqsubseteq \bigsqcup_{1 \leq i \leq l} (D_i \sqcap (\bigsqcap_{1 \leq j \leq l, j \neq i} \neg D_j))$,
9. $D_i \sqsubseteq \forall X. \bigsqcup_{(D_i, D_j) \in \mathcal{H}} D_j \sqcap \forall Y. \bigsqcup_{(D_i, D_k) \in \mathcal{V}} D_k$ *for each* $D_i \in \mathcal{D}$,
10. $A \sqsubseteq \neg B \sqcap \neg C \sqcap \neg D \sqcap \exists X_1^1.B \sqcap \exists Y_1^1.C \sqcap \exists \varepsilon_{AD}.D \sqcap \forall P_{12}^{22}.\bot \sqcap \forall P_{21}^{22}.\bot$,
11. $B \sqsubseteq \neg A \sqcap \neg C \sqcap \neg D \sqcap \exists X_2^2.A \sqcap \exists Y_2^1.D \sqcap \exists \varepsilon_{BC}.C \sqcap \forall P_{21}^{12}.\bot \sqcap \forall P_{12}^{12}.\bot$,
12. $C \sqsubseteq \neg A \sqcap \neg B \sqcap \neg D \sqcap \exists X_2^1.D \sqcap \exists Y_2^2.A \sqcap \exists \varepsilon_{CB}.B \sqcap \forall P_{21}^{21}.\bot \sqcap \forall P_{12}^{21}.\bot$,
13. $D \sqsubseteq \neg A \sqcap \neg B \sqcap \neg C \sqcap \exists X_1^2.C \sqcap \exists Y_1^2.B \sqcap \exists \varepsilon_{DA}.A \sqcap \forall P_{12}^{11}.\bot \sqcap \forall P_{21}^{11}.\bot$.

Theorem 2 (Undecidability of $\mathcal{SHIN}_+$) *The concept* A *is satisfiable w.r.t. concept and role inclusion axioms in Definition 8 iff there is a compatible tiling* t *of the first quadrant* $\mathbb{N} \times \mathbb{N}$ *for a given domino system* $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$.

A proof of Theorem 2 can be found in Appendix.

Conclusion and Discussion

We have presented in this paper a decision procedure for the logic $\mathcal{SHIO}_+$ and shown the finite model property for this logic. To do this we have introduced the neighborhood notion which is an abstraction of the local satisfiability property of tableaux enables us to encapsulate all semantic constraints imposed by the logic constructors in $\mathcal{SHIO}$, and thus to deal with transitive closure of roles independently from the other constructors. With help of this method, we can push the expressiveness of logics to the border between decidability and undecidability.

According to Remark 1, we can devise a decision procedure for deciding the concept satisfiability in $\mathcal{SHIO}_+$ so that it runs in nondeterministic exponential time (NEXPTIME). This result with the proof of Lemma 4 implies that this procedure runs in a deterministic doubly exponential. However, the worst-case complexity of the problem remains an open question.

This work is a crucial step toward an empirical algorithm whose behavior is more goal-directed i.e. the construction of a completion tree would be refined along with satisfying transitive closure of roles, e.g., the non-impacted parts of the tree when rebuilding it would be reused. Such behaviours are inspired from tableaux-based algorithms in which a node of a completion graph should be generated only if necessary.

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Appendix

To prove the results in the report, we need the following definition.

Definition 9 *Let $(\mathcal{T},\mathcal{R})$ be an $\mathcal{SHI}_+$ ontology. A tableau T for a concept D w.r.t $(\mathcal{T},\mathcal{R})$ is defined to be a triplet $(\mathbf{S},\mathcal{L},\mathcal{E})$ such that $\mathbf{S}$ is a set of individuals, $\mathcal{L}$: $\mathbf{S} \to 2^{\mathsf{sub}(\mathcal{T},\mathcal{R},D)}$ and $\mathcal{E}$: $\mathbf{R} \to 2^{\mathbf{S}\times\mathbf{S}}$, and there is some individual $s \in \mathbf{S}$ such that $D \in \mathcal{L}(s)$. For all $s \in \mathbf{S}$, $C, C_1, C_2 \in \mathsf{sub}(\mathcal{T},\mathcal{R},D)$, and R, S, $P^+ \in \mathbf{R}$, T satisfies the following properties:*

- (P1) *If $C_1 \sqsubseteq C_2 \in \mathcal{T}$ and $s \in \mathbf{S}$ then $\mathsf{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}(s)$;*
- (P2) *If $C \in \mathcal{L}(s)$, then $\neg C \notin \mathcal{L}(s)$;*
- (P3) *If $C_1 \sqcap C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ and $C_2 \in \mathcal{L}(s)$;*
- (P4) *If $C_1 \sqcup C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ or $C_2 \in \mathcal{L}(s)$;*
- (P5) *If $\forall S.C \in \mathcal{L}(s)$ and $\langle s,t\rangle \in \mathcal{E}(S)$, then $C \in \mathcal{L}(t)$;*
- (P6) *If $\exists S.C \in \mathcal{L}(s)$, there is $t \in \mathbf{S}$ such that $\langle s,t\rangle \in \mathcal{E}(S)$ and $C \in \mathcal{L}(t)$;*
- (P7) *If $\forall S.C \in \mathcal{L}(s)$, $Q^+ \underline{\circledast} S \in \mathcal{R}$ and $\langle s,t\rangle \in \mathcal{E}(Q^+)$ then $\forall Q^+.C \in \mathcal{L}(t)$;*
- (P8) *$\langle s,t\rangle \in \mathcal{E}(R)$ iff $\langle t,s\rangle \in \mathcal{E}(\mathsf{Inv}(R))$;*
- (P9) *If $\langle s,t\rangle \in \mathcal{E}(P^+)$ then either $\langle s,t\rangle \in \mathcal{E}(P)$, or there exist $t_1,\cdots,t_n \in \mathbf{S}$ such that $\langle s,t_1\rangle,\langle t_1,t_2\rangle,\cdots,\langle t_n,t\rangle \in \mathcal{E}(P)$;*
- (P10) *If $o \in \mathcal{L}(s) \cap \mathcal{L}(t)$ for some $o \in \mathbf{C}_o$ then $s = t$*
- (P11) *If $\langle s,t\rangle \in \mathcal{E}(R)$, $R \underline{\circledast} S$ then $\langle s,t\rangle \in \mathcal{E}(S)$.*

Note that the property P8 in Definition 9 expresses explicitly a cycle for each transitive closure occurring in the label of an edge $\langle s,t\rangle$. A tableau for a concept represents exactly a model for the concept, that is affirmed by the following lemma.

Lemma 5 *An $\mathcal{SHIO}_+$-concept D is satisfiable w.r.t. $(\mathcal{T},\mathcal{R})$ iff D has a tableau.*

Proof of Lemma 5.
• "If-direction". Let $T = (\mathbf{S},\mathcal{L},\mathcal{E})$ be a tableau for D and $D \in \mathcal{L}(s_0)$. A model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ can be defined as follows:

$$\begin{aligned} \Delta^{\mathcal{I}} &= \mathbf{S}, \\ A^{\mathcal{I}} &= \{s \mid A \in \mathcal{L}(s) \text{ for all concept name } A \text{ in } \mathsf{sub}(\mathcal{T},\mathcal{R},D)\}, \\ R^{\mathcal{I}} &= \mathcal{E}(R) \cup \bigcup_{Q^+ \underline{\circledast} R} \mathcal{E}(Q)^+ \text{ for all } R \text{ in } \mathsf{sub}(\mathcal{T},\mathcal{R},D) \end{aligned}$$

To show that $\mathcal{I}$ is a model of D w.r.t. $(\mathcal{T},\mathcal{R})$, we have to show:

1. $\mathcal{I}$ is an interpretation. Indeed,
   (a) Assume $\langle s,t\rangle \in \mathsf{Inv}(R)^{\mathcal{I}}$. By P7 we have $\langle t,s\rangle \in R^{\mathcal{I}}$.
   (b) Assume $\langle s,t\rangle \in R^{\mathcal{I}}$. From the definition of $\mathcal{I}$, there are two cases: (i) there is no $Q^+\underline{\boxast}R \in \mathcal{R}$ i.e. $\langle s,t\rangle \in \mathcal{E}(R)$, (ii) there is $Q^+\underline{\boxast}R \in \mathcal{R}$. This means that there are $\langle s,t_1\rangle,\cdots,\langle t_n,t\rangle \in \mathcal{E}(Q)$. In particular, if $R = Q^+$ i.e. $Q^+\underline{\boxast}R \in \mathcal{R}$ there are $\langle s,t_1\rangle,\cdots,\langle t_n,t\rangle \in \mathcal{E}(Q)$.
   (c) It is easy to check the concept mappings.
2. $\mathcal{I} \models \mathcal{R}$. Assume $R \sqsubseteq S$. We have to prove $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$. Let $\langle s,t\rangle \in R^{\mathcal{I}}$. By P9, it follows $\langle s,t\rangle \in \mathcal{E}(S)$. From the definition of $\mathcal{I}$ we have $\langle s,t\rangle \in S^{\mathcal{I}}$.
3. $\mathcal{I} \models \mathcal{T}$. Assume $C \sqsubseteq D$. We have to prove $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Let $s \in C^{\mathcal{I}}$. From P1 we obtain $\mathsf{nnf}(\neg C \sqcup D) \in \mathcal{L}(s)$. By P2 and P2 we have $s \in D^{\mathcal{I}}$.
4. $D^{\mathcal{I}} \neq \emptyset$.

The last item is proved if we can show that $C \in \mathcal{L}(s)$ implies $s \in C^{\mathcal{I}}$ for all $s \in \mathbf{S}$ (*). In fact, since T is a tableau for D and thus, there exists $s \in \mathbf{S}$ such that $D \in \mathcal{L}(s)$. By (*) it follows $D^{\mathcal{I}} \neq \emptyset$.

We now prove (*) by induction on the length of a concept C, denoted $length(C)$ where C in NNF, is defined as follows:

$$\begin{array}{lllll} len(A) & := & len(\neg A) & := & 0 \\ len(C_1 \sqcap C_2) & := & len(C_1 \sqcup C_2) & := & 1{+}len(C_1) \\ & & & & {+}len(C_2) \\ length(\forall R.C) & := & len(\exists R.C) & := & 1{+}len(C) \end{array}$$

Two basic cases are $C = A$ or $C = \neg A$. If $A \in \mathcal{L}(s)$ then, by the definition of $\mathcal{I}$, $s \in A^{\mathcal{I}}$. If $\neg A \in \mathcal{L}(s)$ then, by P2, $A \notin \mathcal{L}(s)$ and thus $s \notin A^{\mathcal{I}}$. For the inductive step, we have to distinguish several cases:

- $C = C_1 \sqcap C_2$. P3 and $C \in \mathcal{L}(s)$ imply $C_1, C_2 \in \mathcal{L}(s)$. By induction, we have $s \in C_1^{\mathcal{I}}$ and $s \in C_2^{\mathcal{I}}$. Since $\mathcal{I}$ is an interpretation hence $s \in (C_1 \sqcap C_2)^{\mathcal{I}}$.
- $C = C_1 \sqcup C_2$. The same argument.
- $C = \exists S.E$. P6 and $C \in \mathcal{L}(s)$ imply the existence of $t \in \mathbf{S}$ s.t. $\langle s,t\rangle \in \mathcal{E}(S)$ and $E \in \mathcal{L}(t)$. By induction, we have $t \in E^{\mathcal{I}}$ and from the definition of $S^{\mathcal{I}}$, we obtain $\langle s,t\rangle \in S^{\mathcal{I}}$. Since $\mathcal{I}$ is an interpretation hence $s \in (\exists S.E)^{\mathcal{I}} = C^{\mathcal{I}}$.
- $C = \forall S.E$. Let $s \in \mathbf{S}$ with $C \in \mathcal{L}(s)$ and let $t \in \mathbf{S}$ be an individual such that $\langle s,t\rangle \in S^{\mathcal{I}}$. According to the definition of $\mathcal{I}$, we consider the following cases:
  - $\langle s,t\rangle \in \mathcal{E}(S)$. P5 implies $E \in \mathcal{L}(t)$ and by induction, $t \in E^{\mathcal{I}}$. Since $\mathcal{I}$ is an interpretation hence $s \in (\forall S.E)^{\mathcal{I}} = C^{\mathcal{I}}$.
  - $\langle s,t_1\rangle,\cdots,\langle t_n,t\rangle \in \mathcal{E}(Q)$ if there is $Q^+\underline{\boxast}S \in \mathcal{R}$. Since $Q\underline{\boxast}Q^+ \in \mathcal{R}$ we have $\langle s,t_1\rangle,\cdots,\langle t_n,t\rangle \in \mathcal{E}(Q^+)$. Moreover, since $Q^+\underline{\boxast}S$ and P11, we have $\langle s,t_1\rangle,\cdots,\langle t_n,t\rangle \in \mathcal{E}(S)$. By P5 and P7, we have $\{E, \forall Q^+.E\} \subseteq \mathcal{L}(t_i)$ for all $i \in \{1,\cdots,n\}$ and $\{E, \forall Q^+.E\} \subseteq \mathcal{L}(t)$. By induction, $t \in E^{\mathcal{I}}$. Since $\mathcal{I}$ is an interpretation hence $s \in (\forall S.E)^{\mathcal{I}} = C^{\mathcal{I}}$.

• "Only-If-direction". We have to show satisfiability of D w.r.t. $\mathcal{R}$ and $\mathcal{T}$ implies the existence of a tableau T for D w.r.t. $\mathcal{R}$ and $\mathcal{T}$.

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ be a model of D w.r.t. $\mathcal{R}$ and $\mathcal{T}$. A tableau $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ for D can be defined as follows:

$$\begin{array}{rcl} \mathbf{S} & = & \Delta^{\mathcal{I}}, \\ \mathcal{E}(R) & = & R^{\mathcal{I}} \text{ for all } R \text{ occurring in } \mathsf{sub}(\mathcal{T},\mathcal{R},D), \\ \mathcal{L}(s) & = & \{C \in \mathsf{sub}(\mathcal{T},\mathcal{R},D) \mid s \in C^{\mathcal{I}}\} \end{array}$$

- Properties P1, P2, P3, P4, P5 and P6 are obvious.
- Property P7. Assume that $\forall S.C \in \mathcal{L}(s)$ with $Q^+\underline{\boxast}S \in \mathcal{R}$. Let $t \in \mathbf{S}$ be an individual such that $\langle s,t\rangle \in \mathcal{E}(Q^+)$. Assume that there are $\langle t,r_1\rangle,\cdots,\langle r_n,r\rangle \in \mathcal{E}(Q) = Q^{\mathcal{I}}$. We have to show $C \in \mathcal{L}(r)$ since this implies $\forall Q^+.C \in \mathcal{L}(t)$.
  We have $\langle s,t_1\rangle,\cdots,\langle t_m,t\rangle,\langle t,r_1\rangle,\cdots,\langle r_n,r\rangle \in \mathcal{E}(Q) = Q^{\mathcal{I}}$. Due to $Q\underline{\boxast}Q^+ \in \mathcal{R}$, $Q^+\underline{\boxast}S \in \mathcal{R}$ and $\mathcal{I}$ is a model of $\mathcal{R}$, we have $\langle s,r\rangle \in \mathcal{E}(S)$. Moreover, since $\mathcal{I}$ is a model, $\langle s,t\rangle \in \mathcal{E}(S) = S^{\mathcal{I}}$ we have $r \in C^{\mathcal{I}}$. That means that $C \in \mathcal{L}(r)$.
- Property P8 is a consequence of $\mathcal{I} \models \mathcal{R}$ and the definition of $\mathcal{E}$.
- Property P9. Assume that $\langle s,t\rangle \in \mathcal{E}(P^+)$. By the definition of $\mathcal{E}$ we have $\langle s,t\rangle \in (P^+)^{\mathcal{I}}$. This implies that either $\langle s,t\rangle \in P^{\mathcal{I}}$ or there exist $\langle s,t_1\rangle,\cdots,\langle t_n,t\rangle \in P^{\mathcal{I}}$. By the definition of $\mathcal{E}$ we obtain either $\langle s,t\rangle \in \mathcal{E}(P)$ or $\langle s,t_1\rangle,\cdots,\langle t_n,t\rangle \in \mathcal{E}(P)$.
- Property P10. Assume that $\langle s,t\rangle \in \mathcal{E}(R)$ and $R\underline{\boxast}S$. This implies that $\langle s,t\rangle \in R^{\mathcal{I}}$. We consider two cases:
  - $R \sqsubseteq S \in \mathcal{R}$. From $\mathcal{I} \models \mathcal{R}$ it follows $\langle s,t\rangle \in S^{\mathcal{I}}$. By the definition of $\mathcal{E}$ we have $\langle s,t\rangle \in \mathcal{E}(S)$.
  - there exist $R \sqsubseteq S_1 \sqsubseteq,\cdots,\sqsubseteq S_n \sqsubseteq S$. By induction on n it is not hard to show that $\langle s,t\rangle \in S^{\mathcal{I}}$. Thus, by the definition of $\mathcal{E}$ we have $\langle s,t\rangle \in S^{\mathcal{I}}$.
- Property P11. This is deduced from the semantics of nominals : $\mathsf{card}\{o^{\mathcal{I}}\} = 1$ for all $o \in \mathbf{C}_o$.

Lemma (2). *Let D be a $\mathcal{SHIO}_+$-concept w.r.t. a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $G = (V/\sim, E', L)$ be a reduced tableau for D w.r.t. $(\mathcal{T},\mathcal{R})$. We define $\Delta^{\mathcal{I}} = V/\sim$ and a function $\cdot^{\mathcal{I}}$ that maps:*

- *each concept name A occurring in $D, \mathcal{T}$ and $\mathcal{R}$ to $A^{\mathcal{I}} \subseteq V/\sim$ such that $A^{\mathcal{I}} = \{[x] \mid A \in L([x])\}$;*
- *each role name R occurring in $D, \mathcal{T}$ and $\mathcal{R}$ to $R^{\mathcal{I}} \subseteq (V/\sim)^2$ such that $R^{\mathcal{I}} = \{\langle [x],[y]\rangle \mid R \in L\langle [x],[y]\rangle\} \cup \{\langle [y],[x]\rangle \mid \mathsf{Inv}(R) \in L\langle [x],[y]\rangle\}$*

If D has a reduced tableau G then $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a model of D w.r.t. $(\mathcal{T},\mathcal{R})$.

Lemma 2 will be shown if the following is proved.

Lemma 6 *Let D be an $\mathcal{SHI}_+$-concept w.r.t. a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $\mathbf{T} = (V, E, L)$ and $G = (V/\sim, E', L)$ be a completion tree with cyclic paths and tableau graph for D w.r.t. $(\mathcal{T},\mathcal{R})$. We define a triplet $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ from G as follows:*

- $\mathbf{S} = V/\sim$,
- $\mathcal{L}([x]) = L([x])$ *with* $[x] \in V/\sim$,

- $\mathcal{E}(R) = \{\langle [x],[y]\rangle \mid R \in L(\langle [x],[y]\rangle)\} \cup \{\langle [y],[x]\rangle \mid \mathsf{Inv}(R) \in L(\langle [x],[y]\rangle)\}$

It holds that T is a tableau of D w.r.t. $(\mathcal{T},\mathcal{R})$.

Proof of Lemma 6. Let $T = (\mathbf{S},\mathcal{L},\mathcal{E})$. We will prove that T satisfies all the properties from Definition 9.

- $D \in \mathcal{L}([x_0])$ since $D \in L(x_0)$ according to Definition 4;
- Property P1. Let $[x] \in \mathbf{S}$. According to the definition of neighborhood ($tbox$-rule) we have $\mathsf{nnf}(\neg C \sqcup D) \in L(x)$ for all $x \in V$ with $C \sqsubseteq D \in \mathcal{T}$. This implies that $\mathsf{nnf}(\neg C \sqcup D) \in \mathcal{L}(x)$ for all $C \sqsubseteq D \in \mathcal{T}$.
- Property P2 holds since the nodes $x \in V$ are built from valid neighborhoods (i.e. satisfying $clash$-rule) and $\mathcal{L}([x]) = L(x)$.
- Properties P3, P4 hold thanks to the $\sqcap$-rule and $\sqcup$-rule in the definition of neighborhoods (Definition 3);
- Property P5. Assume $\forall S.C \in \mathcal{L}([x])$ and $\langle [x],[y]\rangle \in \mathcal{E}(S)$. According to the definition of $\mathcal{E}$, we consider the following cases:
  1. $S \in L(\langle [x],[y]\rangle)$. By the construction of G, there are $x' \in [x], y' \in [y]$ such that $S \in L(\langle x',y'\rangle)$ or $\mathsf{Inv}(S) \in L(\langle y',x'\rangle)$. By the construction of $\mathbf{T}$ it follows that x', y' are respectively a core and neighbor node of a neighborhood (x',N,l) with $S \in l(x',y')$, $y' \in N$. By $\forall$-rule we have $C \in l(y')$. Moreover, by the construction of $\mathbf{T}$ it follows $C \in L(y')$. By the construction of G, it holds $C \in L([y])$ since $L(y) = L(y')$, and thus $C \in \mathcal{L}([y])$.
  2. $\mathsf{Inv}(S) \in L(\langle [y],[x]\rangle)$. By the construction of G, there are $x' \in [x], y' \in [y]$ such that $\mathsf{Inv}(S) \in L(\langle y',x'\rangle)$ or $S \in L(\langle x',y'\rangle)$. By the construction of $\mathbf{T}$ it follows that x', y' are respectively a core and neighbor node of a neighborhood (y',N,l) with $\mathsf{Inv}(S) \in l(y',x')$, $x' \in N$. By $\forall$-rule we have $C \in l(y')$. Moreover, by the construction of $\mathbf{T}$ it follows $C \in L(y')$. By the construction of G, it holds $C \in L([y])$ since $L(y) = L(y')$ and thus $C \in \mathcal{L}([y])$.
- Property P6. Assume $\exists R.C \in \mathcal{L}([x])$. We will show that there exists $[y] \in \mathbf{S}$ such that $C \in \mathcal{L}([y])$ and $\langle [x],[y]\rangle \in \mathcal{E}(R)$.

  By the construction of G, we have $\exists R.C \in L(x)$. By the construction of $\mathbf{T}$, x is a core node of a neighborhood (x,N,l). By $\exists$-rule there is a neighbor $y \in N$ such that $C \in l(y)$ and $R \in l(\langle x,y\rangle)$. Again, by the construction of $\mathbf{T}$, x has a neighbor y in $\mathbf{T}$ such that $C \in L(y)$ and $R \in L(\langle x,y\rangle)$ or $\mathsf{Inv}(R) \in L(\langle y,x\rangle)$. We consider the following cases:
  - $R \in L(\langle x,y\rangle)$. By the construction of G, we have $C \in L([y])$ and $R \in L(\langle [x],[y]\rangle)$. By the construction of the tableau T, it holds $\langle [x],[y]\rangle \in \mathcal{E}(R)$ and $C \in \mathcal{L}([y])$.
  - $\mathsf{Inv}(R) \in L(\langle y,x\rangle)$. By the construction of G, we have $C \in L([y])$ and $\mathsf{Inv}(R) \in L(\langle [y],[x]\rangle)$. By the construction of the tableau T, it holds $\langle [y],[x]\rangle \in \mathcal{E}(R)$ and $C \in \mathcal{L}([y])$.
- Property P7 is satisfied due to the bidirectional definition of $\mathcal{E}$.
- Property P8. Assume that $\langle [x],[y]\rangle \in \mathcal{E}(Q^+)$ with $Q \in \mathbf{R} \cup \{\mathsf{Inv}(P) \mid P \in \mathbf{R}\}$ and $\langle [x],[y]\rangle \notin \mathcal{E}(Q)$. By the construction of G, there are $x' \in [x], y' \in [y]$ such that $Q^+ \in L(\langle x',y'\rangle)$ and $Q \notin L(\langle x',y'\rangle)$.

  By the construction of $\mathbf{T}$, there is $\varphi = \langle x_0, x_1, x_2, \cdots, x_n, x_{n+1}\rangle$ with $x' = x_2$, and $y' = w$ where $w = x_1$ if x_1 is not blocked or $w = z$ if x_1 is blocked by z. Furthermore, these nodes satisfy the following conditions
  - $L(x_1) = L(x_{n+1}), L(x_0) = L(x_n)$.
  - $Q \in L'(\langle x_i, x_{i+1}\rangle)$ for all $i \in \{2,\cdots,n\}$ where $L'(\langle x_i,x_{i+1}\rangle) = L(\langle x_i,x_{i+1}\rangle)$ if $\langle x_i,x_{i+1}\rangle \in E$; $L'(\langle x_i,x_{i+1}\rangle) = \mathsf{Inv}(\ L(\langle x_{i+1},x_i\rangle)\ )$ if $\langle x_{i+1},x_i\rangle \in E$; $L'(\langle x_i,x_{i+1}\rangle) = L(\langle x_i,z\rangle)\ )$ if $\langle x_i,z\rangle \in E$ and x_{i+1} blocks z.

  By the definition of $\sim$, we have $y', x_{n+1} \in [x_1]$ and $x_0 \in [x_n]$. From the definition of G, we consider the following cases for all $i \in \{2,\cdots,n-1\}$:
  - If $Q \in L(\langle x_i,x_{i+1}\rangle)$ then $Q \in L(\langle [x_i],[x_{i+1}]\rangle)$,
  - If $Q \in \mathsf{Inv}(\ L(\langle x_{i+1},x_i\rangle)\ )$ then $Q \in L(\langle [x_i],[x_{i+1}]\rangle)$,
  - If $Q \in L(\langle x_i,z\rangle)$ where x_{i+1} blocks z then $Q \in L(\langle [x_i],[x_{i+1}]\rangle)$.
  - If $Q \in L(\langle x_n,x_{n+1}\rangle)$ then $Q \in L(\langle [x_n],[x_{n+1}]\rangle)$.

  This implies that $Q \in L(\langle [x_i],[x_{i+1}]\rangle)$ for all $i \in \{2,\cdots,n-1\}$ and $Q \in L(\langle [x_n],[x_1]\rangle)$.
- Property P9. From the construction of neighborhoods. □

Lemma (3). *Let D be an $\mathcal{SHI}_+$-concept. Let $\mathcal{T}$ and $\mathcal{R}$ be a terminology and role hierarchy. If D has a model w.r.t. $(\mathcal{T},\mathcal{R})$ then there exists a completion tree with cyclic paths.*

Proof of Lemma 3.
According to Lemma 5 there is a tableau $T = (\mathbf{S},\mathcal{L},\mathcal{E})$ for D. A tree $\mathbf{T} = (V,E,L)$ can be inductively built from T together with a function π from V to $\mathbf{S}$. This construction is quite intuitive since we can define a valid neighborhood from each individual $s \in \mathbf{S}$ as follows:

- We define $l(v) := \mathcal{L}(s)$. v is valid since any node whose label is included in the label of a node in the tableau T is always valid.
- Let $S'(s) \subseteq \mathbf{S}$ such that $s' \in S'(s)$ iff $\mathcal{L}(\langle s,s'\rangle) \neq \emptyset$ where

  $\mathcal{L}(\langle s,s'\rangle) := \{R \mid \langle s,s'\rangle \in \mathcal{E}(R)$ for some $R \in \mathbf{R}_{(\mathcal{T},\mathcal{R},D)}\}$.

  Let $S(s) \subseteq S'(s)$ such that for each $\mathscr{C} \in 2^{\mathsf{sub}(\mathcal{T},\mathcal{R},D)}$ and $\mathscr{R} \in 2^{\mathbf{R}}$ if there is a $t \in S'(s)$ with $\mathcal{L}(t) = \mathscr{C}$ and $\mathcal{L}(\langle s,t\rangle) = \mathscr{R}$ then there is a unique node $t' \in S(s)$ with $\mathcal{L}(t') = \mathcal{L}(t)$ and $\mathcal{L}(\langle s,t\rangle) = \mathcal{L}(\langle s,t'\rangle)$. This implies that $S(s)$ is finite.
- For each $t \in S(s)$ we define a node $u \in N_0$ such that $l(u) = \mathcal{L}(t)$ and $l(\langle v,u\rangle) = \mathcal{L}(\langle s,s'\rangle)$. From the construction, (v,N_0,l) is valid.

A tree $\mathbf{T} = (V, E, L)$ will be obtained by tiling neighborhoods built from connected individuals started at $s_0 \in \mathbf{S}$ with $D \in \mathcal{L}(s_0)$ and a function π from V to $\mathbf{S}$. Note that if u, v are neighbors in $\mathbf{T}$ then $\pi(u), \pi(v)$ are also neighbors in T. The blocking condition ensures that this construction terminates.

We now build cyclic paths for the transitive closure of roles. By the construction of $\mathbf{T}$, for each $x, y \in V$ such that $Q^+ \in L(\langle x, y\rangle)$, $Q \notin L(\langle x, y\rangle)$ with $Q \in \mathbf{R} \cup \{\mathsf{Inv}(P) \mid P \in \mathbf{R}\}$ we have $\langle \pi(x), \pi(y)\rangle \in \mathcal{E}(Q^+)$.

According to P8 there are $t_1, \cdots, t_n \in \mathbf{S}$ such that $\langle s, t_1\rangle, \cdots, \langle t_n, t\rangle \in \mathcal{E}(Q)$ and $\pi(x) = s, \pi(y) = t$. From this set of edges, we can pick $t_1, \cdots, t_k, t_l, \cdots, t_n$ with $k \leq l$ such that $\mathcal{L}(t_k) = \mathcal{L}(t_l)$ and $\mathcal{L}(t_i) \neq \mathcal{L}(t_j)$ for all $i, j \in \{1, \cdots, k\} \cup \{, l, \cdots, n\}, i \neq j$ (note that $k = l$ if $t_i \neq t_j$ for all $i, j \in \{1, \cdots, n\}, i \neq j$).
We now build a cyclic non-duplicated Q-path from $\{s, t_1, \cdots, t_k, t_l, t_{l+1}, \cdots, t_n, t\}$. Since x is not blocked (x has a successor y), by the construction of $\mathbf{T}$ with $\pi(x) = s$, $\langle s, t_1\rangle \in \mathcal{E}(Q)$, $L(x) = \mathcal{L}(s)$, there exists a neighbor w of x such that $L(w) = \mathcal{L}(t_1)$ and $L'(\langle x, w\rangle) = \mathcal{L}(\langle s, t_1\rangle)$ where $L'(\langle x, w\rangle) = L(\langle x, w\rangle)$. We define $x_1 = w$ and $\pi(w) = t_1$.

Assume that there is $x_i \in V$ (x_i is not blocked by construction) with $\pi(x_i) = t_i$ such that $L(x_i) = \mathcal{L}(t_i)$ with $t_i \in \{t_1, \cdots, t_k, t_l, \cdots, t_n\}$. We consider the following cases:

1. x_i has a neighbor w' such that $L(w') = \mathcal{L}(t_{i+1})$ and $L'(\langle x_i, w'\rangle) = \mathcal{L}(\langle t_i, t_{i+1}\rangle)$ if $i \in \{1, \cdots, k-1, l, \cdots, n-1\}$, or $L(w') = \mathcal{L}(t_{l+1})$ and $L'(\langle x_i, w'\rangle) = \mathcal{L}(\langle t_l, t_{l+1}\rangle)$ if $i = k$. If w' is not blocked then define $x_{i+1} = w'$. If w' is blocked by z then define $x_{i+1} = z$. Since $\langle t_i, t_{i+1}\rangle \in \mathcal{E}(Q)$ hence w' is a Q-neighbor of x_i.
2. x_i has no such a neighbor w'. Since $L(x_i) = \mathcal{L}(t_i)$ if $i \in \{1, \cdots, k-1, l, \cdots, n-1\}$, or $L(x_i) = \mathcal{L}(t_l)$ if $i = k$, by Lemma 1, we can add a successor w' of x_i such that $L(w') = \mathcal{L}(t_{i+1})$ and $L'(\langle x_i, w'\rangle) = \mathcal{L}(\langle t_i, t_{i+1}\rangle)$. If w' is not blocked then define $x_{k+1} = w'$. If w' is blocked by z then define $x_{i+1} = z$. In addition, we define $\pi(w') = t_{i+1}$. Since $\langle t_i, t_{i+1}\rangle \in \mathcal{E}(Q)$ hence w' is a Q-successor of x_i.

Consequently, we obtain $x_1, \cdots, x_k, x_{l+1}, \cdots, x_{n+1}$ such that $L(x_i) = \mathcal{L}(t_i)$ for all $i \in \{1, \cdots, k, l+1, \cdots, n+1\}$; and $Q \in L(\langle x_i, x_{i+1}) = \mathcal{L}(\langle t_i, t_{i+1}\rangle$ for all $i \in \{1, \cdots, k-1, l+1, \cdots, n\}$ or $Q \in L(\langle x_k, x_{l+1}) = \mathcal{L}(\langle t_l, t_{l+1}\rangle)$ with $t_{n+1} = t$.

We define a node w such that $w = y$ if y is not blocked or $w = z$ if y is blocked by z. Since $L(y) = \mathcal{L}(\pi(y)) = \mathcal{L}(t)$ hence $L(x_{n+1}) = \mathcal{L}(t_{n+1}) = \mathcal{L}(t) = L(w)$.

Moreover, we have to find a node z such that z is a $\mathsf{Inv}(Q)$-neighbor of w (w is not blocked). Since $\pi(y) = t, L(w) = L(y)$ hence $L(w) = \mathcal{L}(t)$. We have the following cases: (i) w has a neighbor z such that $L(z) = \mathcal{L}(t_n)$ and $L'(\langle w, z\rangle) = \mathcal{L}(\langle t_t, t_n\rangle)$, (ii) Otherwise, by Lemma 1, we can add a successor z of w such that $L(z) = \mathcal{L}(t_n)$ and $L'(\langle w, z\rangle) = \mathcal{L}(\langle t, t_n\rangle)$. Both cases imply that z is a $\mathsf{Inv}(Q)$-neighbor of w.

According to Definition 4, $\langle w_0, \cdots, w_{n+1}\rangle$ form a cyclic non-duplicated Q-path where $w_0 = z, w_1 = w, w_2 = x$ and $w_i = x_i$ for all $i \in \{2, \cdots, n+1\}$. Thus, $\mathbf{T}$ is a completion tree with cyclic paths for D. □

Lemma (4) **[Termination]** *Algorithm 1 terminates.*

Proof of Lemma 4. Let $m = |\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)|, n = |\mathbf{R}|$ where $|S|$ denotes for the cardinality of a set S. From the construction of completion trees, it holds that:

- Each valid neighborhood has at most $2^{m \times n}$ distinct neighbors. Therefore, we have at most $2^{n(m+1)}$ valid neighborhoods.
- The length of paths from the root to a leaf of a completion tree is bounded by $2^{n(m+1)}$.
- Since the height of completion trees is bounded by the length of paths from the root to a leaf, and each node has at most 2^{mn} neighbors, therefore the number of nodes of a completion tree is bounded by $(2^{mn})^{2^{n(m+1)}}$.
- Since we have at most $2^{n(m+1)}$ valid neighborhoods, hence the number of completion trees is bounded by $\left(2^{n(m+1)}\right)^{(2^{mn})^{2^{n(m+1)}}} = 2^{n(m+1) \times 2^{mn \times 2^{n(m+1)}}}$

In addition, checking whether there is a cyclic path for each occurrence of the transitive closure of a role in the label of an edge is polynomial in the size of completion trees. These facts imply that the algorithm 1 terminates. □

Theorem (2) **[Undecidability of $\mathcal{SHIN}_+$]** *The concept A is satisfiable iff there is a compatible tiling t of the first quadrant $\mathbb{N} \times \mathbb{N}$ for a given domino system $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$.*

Proof of Theorem 2
• *"If-direction"*. Assume that there is a compatible tiling t for $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$. This tiling is used to define an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, .^{\mathcal{I}}\rangle$ of the concept A w.r.t. the axioms in Definition 8. Without loss of the generality, we assume that $t(0,0) = A$. Moreover, each $a_{(m,n)}$ is denoted for each point (m, n) of the first quadrant $\mathbb{N} \times \mathbb{N}$. The figure 1. illustrates the interpretation that we expect.

1. $\Delta^{\mathcal{I}} = \{a_{(m,n)} \mid m, n \in \mathbb{N}\}$
2. $(X_1^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
3. $(X_2^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
4. $(X_2^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
5. $(X_1^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
6. $(Y_1^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
7. $(Y_2^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
8. $(Y_2^1)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$

9. $(Y_1^2)^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
10. $(P_{12}^{11})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
11. $(P_{21}^{11})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
12. $(P_{12}^{22})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
13. $(P_{21}^{22})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
14. $(P_{21}^{21})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
15. $(P_{12}^{21})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
16. $(P_{21}^{12})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
17. $(P_{12}^{12})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\} \cup$ $\{\langle a_{(k,l)}, a_{(k,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
18. $(\varepsilon_{AD})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
19. $(\varepsilon_{DA})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
20. $(\varepsilon_{BC})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
21. $(\varepsilon_{BC})^{\mathcal{I}} = \{\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
22. ${D_i}^{\mathcal{I}} = \{a_{(k,l)} \mid t(k,l) = D_i\}$ for each $D_i \in \mathcal{D}$
23. $A^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 0)\}$
24. $D^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 1)\}$
25. $B^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 1) \wedge (l \bmod 2 = 0)\}$
26. $C^{\mathcal{I}} = \{a_{(k,l)} \mid (k \bmod 2 = 0) \wedge (l \bmod 2 = 1)\}$
27. $X^{\mathcal{I}} = {X_1^1}^{\mathcal{I}} \cup {X_2^1}^{\mathcal{I}} \cup {X_1^2}^{\mathcal{I}} \cup {X_2^2}^{\mathcal{I}}$
28. $Y^{\mathcal{I}} = {Y_1^1}^{\mathcal{I}} \cup {Y_2^1}^{\mathcal{I}} \cup {Y_1^2}^{\mathcal{I}} \cup {Y_2^2}^{\mathcal{I}}$

We now check that $\mathcal{I}$ satisfies all axioms in Definition 8.

1. $X_r^i \sqsubseteq P_{rs}^{ij}, Y_s^j \sqsubseteq P_{rs}^{ij}$ for all $i, j, r, s \in \{1, 2\}, r \neq s$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 2, 6, we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_1^1}^{\mathcal{I}}$, and $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_1^1}^{\mathcal{I}}$. From the assertions 10, 11 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {P_{12}^{11}}^{\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {P_{21}^{11}}^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
2. $X_r^i \sqsubseteq X, Y_r^i \sqsubseteq Y$ for all $i, r \in \{1, 2\}$. From assertions 27 and 28.
3. $\varepsilon_{AD} \sqsubseteq (P_{12}^{11})^+, \varepsilon_{AD} \sqsubseteq (P_{21}^{11})^+$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertion 18, we have $\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \in {\varepsilon_{AD}}^{\mathcal{I}}$. From the assertions 2 and 8 it follows that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_1^1}^{\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {Y_2^1}^{\mathcal{I}}$ (note that $(k + 1 \bmod 2 = 1)$). By the assertion 10 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {P_{12}^{11}}^{\mathcal{I}}$ and $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {P_{12}^{11}}^{\mathcal{I}}$. This implies that $\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \in {(P_{12}^{11})^+}^{\mathcal{I}}$.
     On the other hand, from the assertions 6 and 4 we have $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_1^1}^{\mathcal{I}}$, $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in {X_2^1}^{\mathcal{I}}$ (note that $(l + 1 \bmod 2 = 1)$). By the assertion 11 we have $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {P_{21}^{11}}^{\mathcal{I}}$, $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in {P_{21}^{11}}^{\mathcal{I}}$. This implies that $\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \in {(P_{21}^{11})^+}^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
4. $\varepsilon_{DA} \sqsubseteq (P_{12}^{22})^+, \varepsilon_{DA} \sqsubseteq (P_{21}^{22})^+$. Similarly.
5. $\varepsilon_{BC} \sqsubseteq (P_{21}^{21})^+, \varepsilon_{BC} \sqsubseteq (P_{12}^{21})^+$. Similarly.
6. $\varepsilon_{CB} \sqsubseteq (P_{21}^{12})^+, \varepsilon_{CB} \sqsubseteq (P_{12}^{12})^+$. Similarly.
7. $\top \sqsubseteq\ \leq 1 P_{r,s}^{ij}$ for all $i, j, r, s \in \{1, 2\}, r \neq s$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 10, 11, 14, 17 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {P_{12}^{11}}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {P_{21}^{11}}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {P_{21}^{21}}^{\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {P_{12}^{12}}^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
8. $\top \sqsubseteq\ \leq 1 X, \top \sqsubseteq\ \leq 1 Y$.
   For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 2, 6 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_1^1}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_1^1}^{\mathcal{I}}$. From the assertion 28, we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y^{\mathcal{I}}$.

- Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
- Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
- Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.

9. $\top \sqsubseteq \; \leq 1\varepsilon_{AD}$. It is obvious from the assertion 18 for each $k, l \geq 0$.
10. $\top \sqsubseteq \; \leq 1\varepsilon_{DA}$. It is obvious from the assertion 19 for each $k, l \geq 0$.
11. $\top \sqsubseteq \; \leq 1\varepsilon_{BC}$. It is obvious from the assertion 20 for each $k, l \geq 0$.
12. $\top \sqsubseteq \; \leq 1\varepsilon_{CB}$. It is obvious from the assertion 21 for each $k, l \geq 0$.
13. $\top \sqsubseteq \bigsqcup_{1\leq i\leq l} (D_i \sqcap (\bigsqcap_{1\leq j\leq l, j\neq i} \neg D_j))$. Since t is a tiling, each (k,l) has a unique $D_i \in \mathcal{D}$ such that $t(k,l) = D_i$. Thus, from the assertion 22, each $a_{(k,l)}$ has a unique $D_i \in \mathcal{D}$ such that $a_{(k,l)} \in {D_i}^{\mathcal{I}}$.
14. $D_i \sqsubseteq \forall X. \bigsqcup_{(D_i,D_j)\in\mathcal{H}} D_j \sqcap \forall Y. \bigsqcup_{(D_i,D_k)\in\mathcal{V}} D_k$ for each $D_i \in \mathcal{D}$.
From the assertion 22, if $a_{(k,l)} \in {D_i}^{\mathcal{I}}$ then $t(k,l) = D_i$. Since t is a tiling, according to Definition 7 we have $\langle D_i, D_j\rangle \in \mathcal{H}$ and $\langle D_i, D_k\rangle \in \mathcal{V}$ with $t(k+1,l) = D_j$ and $t(k,l+1) = D_k$. From the assertions 28 and 2-9 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in X^{\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in Y^{\mathcal{I}}$. From the assertion 22, we have $a_{(k+1,l)} \in {D_j}^{\mathcal{I}}$ and $a_{(k,l+1)} \in {D_k}^{\mathcal{I}}$.
15. $A \sqsubseteq \neg B \sqcap \neg C \sqcap \neg D \sqcap \exists X_1^1.B \sqcap \exists Y_1^1.C \sqcap \exists\varepsilon_{AD}.D \sqcap \forall P_{12}^{22}.\bot \sqcap \forall P_{21}^{22}.\bot$.
For each $k, l \geq 0$, we consider the following cases:
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. From the assertions 23 we have $a_{(k,l)} \in A^{\mathcal{I}}$. From the assertions 24, 25, 26, we have $a_{(k,l)} \notin B^{\mathcal{I}}, a_{(k,l)} \notin C^{\mathcal{I}}, a_{(k,l)} \notin D^{\mathcal{I}}$. Moreover, from the assertions 2, 6 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_1^1}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_1^1}^{\mathcal{I}}$. By the assertions 18 and 24 we have $\langle a_{(k,l)}, a_{(k+1,l+1)}\rangle \in {\varepsilon_{AD}}^{\mathcal{I}}$ and $a_{(k+1,l+1)} \in D^{\mathcal{I}}$.
   Additionally, according to the assertions 12, 13, $\langle a_{(k,l)}, a_{(k+1,l)}\rangle$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \notin {P_{12}^{11}}^{\mathcal{I}}$ and $\langle a_{(k,l)}, a_{(k+1,l)}\rangle, \langle a_{(k,l)}, a_{(k,l+1)}\rangle \notin {P_{21}^{11}}^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. From the assertion 23, it follows $a_{(k,l)} \notin A^{\mathcal{I}}$.
   - Assume $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly.
   - Assume $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
16. $B \sqsubseteq \neg A \sqcap \neg C \sqcap \neg D \sqcap \exists X_2^2.A \sqcap \exists Y_2^1.D \sqcap \exists\varepsilon_{BC}.C \sqcap \forall P_{21}^{12}.\bot \sqcap \forall P_{12}^{12}.\bot$. Similarly.
17. $C \sqsubseteq \neg A \sqcap \neg B \sqcap \neg D \sqcap \exists X_2^1.D \sqcap \exists Y_2^2.A \sqcap \exists\varepsilon_{CB}.B \sqcap \forall P_{21}^{21}.\bot \sqcap \forall P_{12}^{21}.\bot$. Similarly.
18. $D \sqsubseteq \neg A \sqcap \neg B \sqcap \neg C \sqcap \exists X_1^2.C \sqcap \exists Y_1^2.B \sqcap \exists\varepsilon_{DA}.A \sqcap \forall P_{12}^{11}.\bot \sqcap \forall P_{21}^{11}.\bot$. Similarly.

- *"Only-If-direction"*. On the other hand, assume that the concept A is satisfiable w.r.t. the axioms in Definition 8 , and let $\mathcal{I} = \langle\Delta^{\mathcal{I}}, .^{\mathcal{I}}\rangle$ be an interpretation such that $A^{\mathcal{I}} \neq \emptyset$. Assume that $a_{(0,0)} \in A^{\mathcal{I}}$. This interpretation can be used to find a compatible tiling for $\mathbf{D}$.

First, we show the following claim:

Claim 1 *There are individuals $a_{(k,l)} \in \Delta^{\mathcal{I}}$ with $k, l \geq 0$ such that*

- *If $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$ then $a_{(k,l)} \in A^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in B^{\mathcal{I}}, a_{(k,l+1)} \in C^{\mathcal{I}}, a_{(k+1,l+1)} \in D^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_1^1}^{\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {Y_2^1}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_1^1}^{\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in {X_2^1}^{\mathcal{I}}$.*
- *If $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$ then $a_{(k,l)} \in D^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in C^{\mathcal{I}}, a_{(k,l+1)} \in B^{\mathcal{I}}, a_{(k+1,l+1)} \in A^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_1^2}^{\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {Y_2^2}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_1^2}^{\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in {X_2^2}^{\mathcal{I}}$.*
- *If $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$ then $a_{(k,l)} \in B^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in A^{\mathcal{I}}, a_{(k,l+1)} \in D^{\mathcal{I}}, a_{(k+1,l+1)} \in C^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_2^2}^{\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {Y_1^1}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_2^1}^{\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in {X_1^2}^{\mathcal{I}}$.*
- *If $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$ then $a_{(k,l)} \in C^{\mathcal{I}}$. Additionally, there are $a_{(k+1,l)} \in D^{\mathcal{I}}, a_{(k,l+1)} \in A^{\mathcal{I}}, a_{(k+1,l+1)} \in B^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X_2^1}^{\mathcal{I}}$, $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {Y_1^2}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y_2^2}^{\mathcal{I}}$ and $\langle a_{(k,l+1)}, a_{(k+1,l+1)}\rangle \in {X_1^1}^{\mathcal{I}}$.*

Proof:[Proof of the claim 1]

- Assume $k = 0, l = 0$. We have $a_{(0,0)} \in A^{\mathcal{I}}$. By the axiom 10 in Definition 8 there are $a_{(1,0)} \in B^{\mathcal{I}}, a_{(0,1)} \in C^{\mathcal{I}}$ such that $\langle a_{(0,0)}, a_{(1,0)}\rangle \in {X_1^1}^{\mathcal{I}}$, $\langle a_{(0,0)}, a_{(0,1)}\rangle \in {Y_1^1}^{\mathcal{I}}$. Moreover, by the axioms 11, 12 in Definition 8 there are $a_{(1,1)}, a'_{(1,1)} \in D^{\mathcal{I}}$ such that $\langle a_{(1,0)}, a_{(1,1)}\rangle \in {Y_2^1}^{\mathcal{I}}$, $\langle a_{(0,1)}, a'_{(1,1)}\rangle \in {X_2^1}^{\mathcal{I}}$. We show that $a'_{(1,1)} = a_{(1,1)}$.
By the axiom 10 in Definition 8, let $a \in D^{\mathcal{I}}$ such that $\langle a_{(0,0)}, a\rangle \in \varepsilon_{AD}^{\mathcal{I}}$. From the axiom 1 in Definition 8 we have $\langle a_{(0,0)}, a_{(1,0)}\rangle, \langle a_{(1,0)}, a_{(1,1)}\rangle \in {P_{12}^{11}}^{\mathcal{I}}$. If $a_{(1,1)} \neq a$ then, by the axioms 3, 5 in Definition 8 there is an instance a' such that $\langle a_{(1,1)}, a'\rangle \in {P_{12}^{11}}^{\mathcal{I}}$, which contradicts the axiom 13 in Definition 8 since $a_{(1,1)} \in D^{\mathcal{I}}$ and $\langle a_{(1,1)}, a'\rangle \in {P_{12}^{11}}^{\mathcal{I}}$. Thus, $a_{(1,1)} = a$. Analogously, from the axiom 1 in Definition 8 we have $\langle a_{(0,0)}, a_{(0,1)}\rangle, \langle a_{(0,1)}, a'_{(1,1)}\rangle \in {P_{21}^{11}}^{\mathcal{I}}$. If $a'_{(1,1)} \neq a$ then, by the axioms 3, 5 in Definition 8 there is an instance a'' such that $\langle a'_{(1,1)}, a''\rangle \in {P_{21}^{11}}^{\mathcal{I}}$, which contradicts the axiom 13 in Definition 8 since $a'_{(1,1)} \in D^{\mathcal{I}}$ and

$\langle a'_{(1,1)}, a''\rangle \in {P^{11}_{21}}^{\mathcal{I}}$. Therefore, $a'_{(1,1)} = a$, and thus $a_{(1,1)} = a'_{(1,1)}$.

- Assume that $k \geq 0$ or $l \geq 0$. We consider the following cases:
  - Assume $a_{(k,l)} \in A^{\mathcal{I}}$ with $(k \bmod 2 = 0) \wedge (l \bmod 2 = 0)$. By the axiom 10 in Definition 8 there are $a_{(k+1,l)} \in B^{\mathcal{I}}, a_{(k,l+1)} \in C^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a_{(k+1,l)}\rangle \in {X^1_1}^{\mathcal{I}}$, $\langle a_{(k,l)}, a_{(k,l+1)}\rangle \in {Y^1_1}^{\mathcal{I}}$. Moreover, by the axioms 11, 12 in Definition 8 there are $a_{(k+1,l+1)}, a'_{(k+1,l+1)} \in D^{\mathcal{I}}$ such that $\langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {Y^1_2}^{\mathcal{I}}$, $\langle a_{(k,l+1)}, a'_{(k+1,l+1)}\rangle \in {X^1_2}^{\mathcal{I}}$. We show that $a'_{(k+1,l+1)} = a_{(k+1,l+1)}$.
    By the axiom 10 in Definition 8, let $a \in D^{\mathcal{I}}$ such that $\langle a_{(k,l)}, a\rangle \in \varepsilon^{\mathcal{I}}_{AD}$. From the axiom 1 in Definition 8 we have $\langle a_{(k,l)}, a_{(k+1,l)}\rangle, \langle a_{(k+1,l)}, a_{(k+1,l+1)}\rangle \in {P^{11}_{12}}^{\mathcal{I}}$. If $a_{(k+1,l+1)} \neq a$ then, by the axioms 3, 5 in Definition 8 there is an instance a' such that $\langle a_{(k+1,l+1)}, a'\rangle \in {P^{11}_{12}}^{\mathcal{I}}$, which contradicts the axiom 13 in Definition 8 since $a_{(k+1,l+1)} \in D^{\mathcal{I}}$ and $\langle a_{(k+1,l+1)}, a'\rangle \in {P^{11}_{12}}^{\mathcal{I}}$. Thus, $a_{(k+1,l+1)} = a$. Analogously, from the axiom 1 in Definition 8 we have $\langle a_{(k,l)}, a_{(k,l+1)}\rangle, \langle a_{(k,l+1)}, a'_{(k+1,l+1)}\rangle \in {P^{11}_{21}}^{\mathcal{I}}$. If $a'_{(k+1,l+1)} \neq a$ then, by the axioms 3, 5 in Definition 8 there is an instance a'' such that $\langle a'_{(k+1,l+1)}, a''\rangle \in {P^{11}_{21}}^{\mathcal{I}}$, which contradicts the axiom 13 in Definition 8 since $a'_{(k+1,l+1)} \in D^{\mathcal{I}}$ and $\langle a'_{(k+1,l+1)}, a''\rangle \in {P^{11}_{21}}^{\mathcal{I}}$. Therefore, $a'_{(k+1,l+1)} = a$, and thus $a_{(k+1,l+1)} = a'_{(k+1,l+1)}$.
    Obviously, if $(k \bmod 2 = 0)$ and $(l \bmod 2 = 0)$ then $((k+1) \bmod 2 = 1)$ and $((l+1) \bmod 2 = 1)$
  - Assume $a_{(k,l)} \in D^{\mathcal{I}}$ with $(k \bmod 2 = 1) \wedge (l \bmod 2 = 1)$. Similarly.
  - Assume $a_{(k,l)} \in B^{\mathcal{I}}$ with $(k \bmod 2 = 1) \wedge (l \bmod 2 = 0)$. Similarly.
  - Assume $a_{(k,l)} \in C^{\mathcal{I}}$ with $(k \bmod 2 = 0) \wedge (l \bmod 2 = 1)$. Similarly. □

□

We now define a mapping $t : \mathbb{N} \times \mathbb{N} \to \mathcal{D}$ as follows. By the axiom 8 in Definition 8, there is $D_i \in \mathcal{D}$ such that $a_{(0,0)} \in {D_i}^{\mathcal{I}}$.

1. $t(0,0) := D_i$ with $a_{(0,0)} \in {D_i}^{\mathcal{I}}$. From the axioms 9, 2, 6 in Definition 8 and Claim 1, there are $D^{(0,0)}_x, D^{(0,0)}_y \in \mathcal{D}$ such that $(D_i, D_x) \in \mathcal{H}$, $(D_i, D^{(0,0)}_y) \in \mathcal{V}$, and $\langle a_{(0,0)}, a_{(1,0)}\rangle \in X^{\mathcal{I}}$ with $a_{(1,0)} \in {D^{(0,0)}_x}^{\mathcal{I}}$, $\langle a_{(0,0)}, a_{(0,1)}\rangle \in Y^{\mathcal{I}}$ with $a_{(0,1)} \in {D^{(0,0)}_y}^{\mathcal{I}}$. Therefore, we define $t(1,0) := D^{(0,0)}_x, t(0,1) := D^{(0,0)}_y$. Since X, Y are functional and D_h are disjoint for all $D_h \in \mathcal{D}$ hence such $D^{(0,0)}_x, D^{(0,0)}_y$ are uniquely determined from D_i.
   Moreover, from the axiom 9, 2, 6 in Definition 8, there are $D^{(1,0)}_y, D^{(0,1)}_x \in \mathcal{D}$ such that $(D^{(0,0)}_x, D^{(1,0)}_y) \in \mathcal{H}$, $(D^{(0,0)}_y, D^{(0,1)}_x) \in \mathcal{V}$, and $\langle a_{(1,0)}, a_{(1,1)}\rangle \in Y^{\mathcal{I}}$ with $a_{(1,1)} \in {D^{(1,0)}_y}^{\mathcal{I}}$, $\langle a_{(0,1)}, a'_{(1,1)}\rangle \in X^{\mathcal{I}}$ with $a'_{(1,1)} \in {D^{(0,1)}_x}^{\mathcal{I}}$. By the axioms 11, 12, 2, 6 in Definition 8 we have $\langle a_{(1,0)}, a_{(1,1)}\rangle \in {Y^1_2}^{\mathcal{I}}$, $\langle a_{(0,1)}, a'_{(1,1)}\rangle \in {X^1_2}^{\mathcal{I}}$. From Claim 1 we have $a_{(1,1)} = a'_{(1,1)}$. This implies that $D^{(1,0)}_y = D^{(0,1)}_x$ since $D^{(1,0)}_y, D^{(0,1)}_x$ are disjoint by the axiom 8 in Definition 8. Therefore we can define $t(1,1) := D^{(1,0)}_y = D^{(0,1)}_x$.
2. Assume that $t(i,j) := D_{i'}$ with $a(i,j) \in {D_{i'}}^{\mathcal{I}}$. From the axiom 9, 2, 6 in Definition 8 and Claim 1, there are $D^{(i,j)}_x, D^{(i,j)}_y \in \mathcal{D}$ such that $(D^{(i,j)}_x, D^{(i,j)}_y) \in \mathcal{H}$, $(D^{(i,j)}_x, D^{(i,j)}_y) \in \mathcal{V}$, and $\langle a_{(i,j)}, a_{(i+1,j)}\rangle \in X^{\mathcal{I}}$ with $a_{(i+1,j)} \in {D^{(i,j)}_x}^{\mathcal{I}}$, $\langle a_{(i,j)}, a_{(i,j+1)}\rangle \in Y^{\mathcal{I}}$ with $a_{(i,j+1)} \in {D^{(i,j)}_y}^{\mathcal{I}}$. Therefore, $t(i+1,j) := D^{(i,j)}_x, t(i,j+1) := D^{(i,j)}_y$. Since X, Y are functional and D_h are disjoint for all $D_h \in \mathcal{D}$ hence such $D^{(i,j)}_x, D^{(i,j)}_y$ are uniquely determined from $D_{i'}$.
   Moreover, from the axiom 9, 2, 6 in Definition 8, there are $D^{(i+1,j)}_y, D^{(i,j+1)}_x \in \mathcal{D}$ such that $(D^{(i,j)}_x, D^{(i+1,j)}_y) \in \mathcal{H}$, $(D^{(i,j)}_y, D^{(i,j+1)}_x) \in \mathcal{V}$, and $\langle a_{(i+1,j)}, a_{(i+1,j+1)}\rangle \in Y^{\mathcal{I}}$ with $a_{(i+1,j+1)} \in {D^{(i+1,j)}_y}^{\mathcal{I}}$, $\langle a_{(i,j+1)}, a'_{(i+1,j+1)}\rangle \in X^{\mathcal{I}}$ with $a'_{(i+1,j+1)} \in {D^{(i,j+1)}_x}^{\mathcal{I}}$. We now distinguish the following cases:
   (a) Assume that $a_{(i,j)} \in A^{\mathcal{I}}$. From Claim 1 and the axiom 8 in Definition 8 we can show $D^{(i+1,j)}_y = D^{(i,j+1)}_x$. Therefore we can define $t(i+1,j+1) := D^{(i+1,j)}_y = D^{(i,j+1)}_x$.
   (b) Assume that $a_{(i,j)} \in B^{\mathcal{I}}$. Similarly.
   (c) Assume that $a_{(i,j)} \in C^{\mathcal{I}}$. Similarly.
   (d) Assume that $a_{(i,j)} \in D^{\mathcal{I}}$. Similarly.

It remains to be shown that (1) t is well defined, (2) the horizontal and vertical matching conditions are satisfied.

(1) is obvious from the construction of the mapping t.

(2) From the definition of t, for each $a_{(k,l)}$ there is a $D_i \in \mathcal{D}$ such that $t(k,l) = D_i$ and $a_{(k,l)} \in {D_i}^{\mathcal{I}}$. Again, by the construction of t, there are $D_j, D_k \in \mathcal{D}$ such that $t(k+1,l) = D_j$, $t(k,l+1) = D_j$ and $a_{(k+1,l)} \in {D_j}^{\mathcal{I}}$, $a_{(k,l+1)} \in {D_k}^{\mathcal{I}}$. By the axioms 2 and 9, we have $\langle D_i, D_j\rangle \in \mathcal{H}$ and $\langle D_i, D_k\rangle \in \mathcal{V}$.